STRONG MATCHING PRECLUSION FOR AUGMENTED CUBES

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Abstract. The strong matching preclusion number of a graph is the minimum number of vertices and edges whose deletion results in a graph that has neither perfect matchings nor almost perfect matchings. The concept was introduced by Park and Son. In this paper, we study the strong matching preclusion problem for the augmented cube graphs. As a result, we find $\mathrm{smp}(AQ_n)$ and classify all optimal solutions.

Keywords: Interconnection networks, perfect matching, augmented cubes

1. Introduction

A *perfect matching* in a graph is a set of edges such that every vertex is incident with exactly one edge in this set. An *almost perfect matching* in a graph is a set of edges such that every vertex except one is incident with exactly one edge in this set, and the exceptional vertex is incident to none. If a graph has a perfect matching, then it has an even number of vertices; if a graph has an almost perfect matching, then it has an odd number of vertices. The *matching preclusion number* of a graph G, denoted by $\mathrm{mp}(G)$, is the minimum number of edges whose deletion leaves the resulting graph without a perfect matching or an almost perfect matching. Any such optimal set is called an *optimal matching preclusion set*. If G has neither a perfect matching nor an almost perfect matching, then $\mathrm{mp}(G) = 0$. This concept of matching preclusion was introduced by [1] and further studied by [4–10,23,24,26]. They introduced this concept as a measure of robustness in the event of edge failure in interconnection networks, as well as a theoretical connection to conditional connectivity, "changing and unchanging of invariants" and extremal graph theory. We refer the readers

Date: March 15, 2013.

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$^{\alpha}$A project started at the 2011 Oakland University Summer Mathematics Institute under the supervision of E. Cheng. A preliminary version of this project received an United States Army Award and an United Stated Air Force Certificate of Achievement at the 2012 Detroit Science Fair.

to [1] for details and additional references. In [25], the concept of strong matching preclusion was introduced. The *strong matching preclusion number* of a graph G, denoted by $\text{smp}(G)$, is the minimum number of vertices and edges whose deletion leaves the resulting graph without a perfect matching or an almost perfect matching. Any such optimal set is called an *optimal strong matching preclusion set*.

Useful distributed processor architectures offer the advantages of improved connectivity and reliability. An important component of such a distributed system is the system topology, which defines the inter-processor communication architecture. Such system topology forms the interconnection network. We refer the readers to [16] for recent progress in this area and the references in its extensive bibliography. In certain applications, every vertex requires a special partner at any given time and the matching preclusion number measures the robustness of this requirement in the event of link failures as indicated in [1]. Hence in these interconnection networks, it is desirable to have the property that the only optimal matching preclusion sets and optimal strong matching preclusion sets are those whose deletion gives an isolated vertex in the resulting graph. Since interconnection networks are usually even, we only consider even graphs in this paper, that is, graphs with even number of vertices.

Proposition 1.1. *Let G be a graph with an even number of vertices. Then $smp(G) \leq mp(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree of G.*

Proof. Since G is even, $\text{mp}(G)$ is the minimum number of edges whose deletion leaves a graph with no perfect matchings. Since deleting all edges incident to a single vertex will give a graph with no perfect matchings, $\text{mp}(G) \leq \delta(G)$. The claim $\text{smp}(G) \leq \text{mp}(G)$ is obviously true as every matching preclusion set is a strong matching preclusion set. □

An optimal solution of the form given in the proof of Proposition 1.1 is a *trivial (optimal) matching preclusion set*. Let F be an optimal strong matching preclusion set of a graph $G = (V, E)$. Suppose $F = F_V \cup F_E$ where F_V consists of vertices and F_E consists of edges. We may assume that no element in F_E is incident to an element in F_V since F is optimal. (If $e \in F_E$ is incident to an element of F_V, then $G - F = G - (F - \{e\})$.) We call F a *basic (optimal) strong matching preclusion set* if F is an optimal strong matching preclusion set of G and $G - F$ has an isolated vertex, that is, there exists a vertex v such that every vertex in F_V is a neighbor of v and every edge in F_E is incident to v. This includes the following scenario: F is a basic optimal matching preclusion set and $G - F$ is odd without almost perfect matchings. We can further restrict this class as follows: If $G - F$ is even and there is a vertex v such that every vertex in F_V is a neighbor of v and every edge in F_E is incident to v, then F is a *trivial (optimal) strong matching preclusion set.* For r-regular even graphs we have the following relationship between these classes of preclusion sets.

Proposition 1.2 ([2])**.** *Let $r \geq 2$. Let G be an r-regular even graph. Suppose that $smp(G) = r$. Then every basic strong matching preclusion set is trivial.*

Hypercubes are the most basic class of interconnection networks. However, they have shortcomings and a number of their variants were introduced to address some of the issues. One such popular variant is the class of augmented cubes introduced in [11]. As an improvement upon the hypercubes, the augmented cube graphs are designed to be superior in many aspects. Not only do they retain some of the favorable properties of the hypercubes but also possess some embedding properties that the hypercubes do not have. For instance, a hypercube of the n^{th} dimension contains cycles of all lengths from 3 to 2^n whereas the hypercube contains only even cycles. As shown in [25], bipartite graphs are poor interconnection networks with respect to the strong matching preclusion property. However, augmented cubes are not bipartite and we will show in this paper that they have good strong matching preclusion properties.

We now define the n-dimensional augmented cube AQ_n as follows. Let $n \geq 1$, the graph AQ_n has 2^n vertices, each labeled by an n-bit binary string $u_1 u_2 \cdots u_n$ such that $u_i \in \{0, 1\}$ for all i. AQ_1 is isomorphic to the complete graph K_2 where one vertex is labeled by the digit 0 and the other by 1. For $n \geq 2$, AQ_n is defined recursively by using two copies of $(n-1)$-dimensional augmented cubes with edges between them. We first add the digit 0 to the beginning of the binary strings of all vertices in one copy of AQ_{n-1}, which will be denoted by AQ^0_{n-1}, and add the digit 1 to the beginning of all the vertices of the second copy, which will be denoted by AQ^1_{n-1}. We now describe the edges between these two copies. Let $u = 0u_1 u_2 \cdots u_{n-1}$ and $v = 1v_1 v_2 \cdots v_{n-1}$ be vertices in AQ^0_{n-1} and AQ^1_{n-1}, respectively. Then u and v are adjacent if and only if one of the following conditions holds:

(1) $u_i = v_i$ for every $i \geq 1$. In this case, we call the edge (u, v) a *cross edge* and say $u = v^x$ and $v = u^x$.

(2) $u_i \neq v_i$ for every $i \geq 1$. In this case, we call (u, v) a *complement edge* and denote $u = v^c$ and $v = u^c$.

Throughout this paper, we denote the set of cross edges and complement edges in AQ_n by X_n and C_n respectively. Clearly, AQ_n is $(2n-1)$-regular, $|C_n| = |X_n| = 2^{n-1}$ and the edges in C_n (X_n) are independent. It is well-known that AQ_n is vertex-transitive. Another important fact is that the connectivity of AQ_n is $2n-1$ for $n \geq 4$. Some recent papers on augmented cubes include [3, 6, 13–15, 17, 21, 22]. A few examples of augmented cubes are shown in Figure 1.

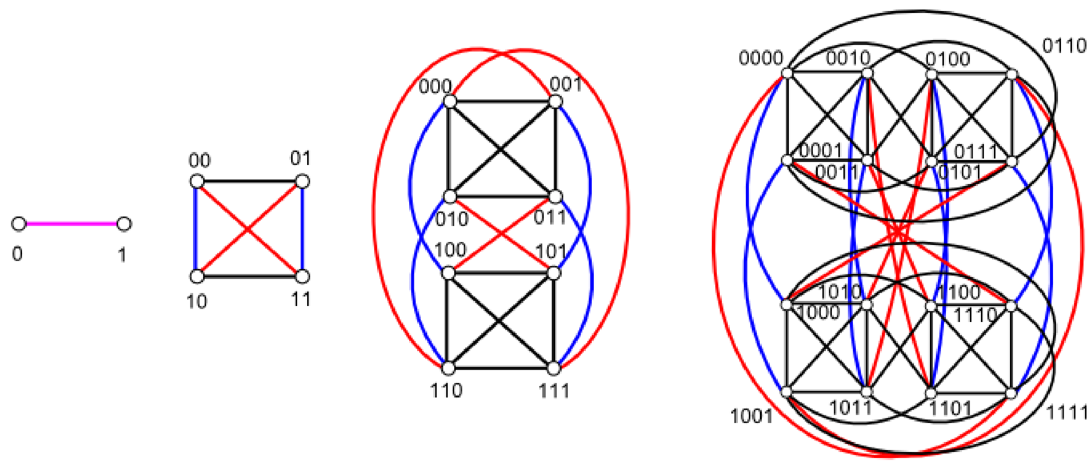

Figure 1.1. Augmented cubes of dimensions 1 through 4

2. Preliminaries

Our objective is to show that $\operatorname{smp}(AQ_n) = 2n-1$, which is the best possible result, and that all optimal solutions are trivial. In this section, we present some results that will be useful in our quest. Since the strong matching preclusion problem is a generalization of the matching preclusion problem and the latter problem has been solved for AQ_n, we state the corresponding result.

Theorem 2.1 ([6]). *Suppose $n \geq 4$. Then $mp(AQ_n) = 2n - 1$. Moreover, every optimal matching preclusion set is trivial.*

Given that a Hamilton cycle in an even graph induces two edge-disjoint perfect matchings, the following result uses "fault Hamiltonian" property as a sufficient condition in determining the strong matching preclusion number.

Proposition 2.2 ([2]). *Let G be an r-regular even graph with the property that $G - F$ is Hamiltonian for every $F \subseteq V(G) \cup E(G)$ where $|F| \leq r - 2$. Then $smp(G) = mp(G) = r$.*

However, we are unaware of any relationship between such "fault Hamiltonian" property and the classification of optimal strong matching preclusion sets. In order to apply Proposition 2.2, we need Hamiltonian results for AQ_n. Fortunately, such a result is known.

Theorem 2.3 ([15]). *Let $n \geq 4$. Suppose $F \subseteq V(AQ_n) \cup E(AQ_n)$. If $|F| \leq 2n - 4$, then $AQ_n - F$ is Hamiltonian connected* [1]*; if $|F| \leq 2n - 3$, then $AQ_n - F$ is Hamiltonian.*

3. Main Result

It follows from Proposition 2.2 and Theorem 2.3 that $\mathrm{smp}(AQ_n) = 2n - 1$ for $n \geq 4$. It remains to classify all optimal solutions. We claim that all optimal solutions are trivial. Given the recursive structure of augmented cubes, the natural method is to use induction. The first step is to check the base case.

Lemma 3.1. *$smp(AQ_4) = 7$. Moreover every optimal strong matching preclusion set is trivial.*

Proof. This result was verified by a computer program. The check was done using a Python program and the NetworX package [12] for graph representation. The program verified that for every 7-element fault sets F, unless F is trivial, $AQ_4 - F$ has a perfect matching or an almost perfect matching. In order to reduce the number of cases that had to be checked we note that Theorem 2.1 implies that that we may assume F contains at least one vertex. Moreover, since AQ_4 is vertex-transitive, one vertex in F can be fixed. Additionally, it can be

[1] A graph is Hamiltonian connected if there is a Hamiltonian path between every pair of vertices.

assumed that no fault edge is incident with a fault vertex. In wall clock time, the computer verification took a couple days on a modern desktop computer. □

Before we present the proof of our main result, we need a number of easy technical results. We start with the following useful observation of augmented cubes which we will apply without explicitly referencing it.

Proposition 3.2. *Let $n \geq 3$. Let u be a vertex of AQ_n. Then u^x is adjacent to u^c. Moreover, there is a unique vertex v such that u and v are adjacent, $v^c = u^x$ and $v^x = u^c$. In other words, u, v, u^x, u^c form a complete graph on four vertices.*

We need two more facts regarding matchings which we will now state without proof.

Proposition 3.3. *Let G be a graph with no isolated vertices. Suppose that G has an almost perfect matching M that misses vertex v. Then there exists an almost perfect matching in G which misses a vertex other than v.*

Proposition 3.4. *Let G be a graph with no isolated vertices. Suppose that G has an almost-perfect matching M that misses vertex w. If G does not contain a 2-path $v - u - w$ in which v and w have degree 1, then there exist almost-perfect matchings M_1 and M_2 in G such that M, M_1 and M_2 miss different vertices.*

Our main result is that every optimal conditional strong matching preclusion set in AQ_n is trivial. Before proceeding with the proof we give some comments on the general strategy that will applied. Due to the recursive structure of AQ_n it is natural to establish the result by induction. In particular, given any fault set F that is not a trivial strong matching preclusion set, we must show how to construct a perfect matching or almost perfect matching in $AQ_n - F$. We will consider several cases regarding how the faults are distributed among AQ^0_{n-1}, AQ^1_{n-1} and the set of cross edges. If many faults are concentrated within one of these two subgraphs, the induction hypothesis cannot be directly applied to recover a perfect matching or almost perfect matching in that subgraph. In such cases we will remove a set A from the fault set so that induction can be applied, building a perfect matching or almost perfect matching in each of the subgraphs and finally using the structural properties of AQ_n to augment the matchings to form a perfect matching or almost perfect matching in the entire graph, removing any dependence on the set A. Finally, even if the faults are distributed more evenly between AQ_{n-1}^0 and AQ_{n-1}^1 some care must still be taken; if an odd number of fault vertices appear in each half then induction will only provide an almost perfect matching in each, the union of these must still be augmented to produce a perfect matching in the entire graph. The art is to find the right balance in dividing the cases. This type of case analysis is the frequent method of choice in this area as seen in [18, 20, 24–26] among others for matching preclusion. Proofs for other properties on interconnection networks are equally involved; for example, see [13–15, 19, 21].

Theorem 3.5. *Let $n \geq 4$. Then smp$(AQ_n) = 2n - 1$. Moreover, every optimal strong matching preclusion set is trivial.*

Proof. The claim that $\text{smp}(AQ_n) = 2n - 1$ follows from Proposition 2.2 and Theorem 2.3. We now classify the optimal solutions. The proof is via induction. We first note that the statement is true if $n = 4$ by Lemma 3.1. Let $n \geq 5$ and assume that the result is true for AQ_{n-1}. Let $F \subseteq V(AQ_n) \cup E(AQ_n)$ be an optimal strong matching preclusion set. As remarked earlier, we may assume that no edge in F is incident to a vertex in F. Then, we show that either $AQ_n - F$ contains a perfect matching or an almost perfect matching, or that F is a trivial strong matching preclusion set of AQ_n. Let $F = F_X \cup F_C \cup F_0 \cup F_1$ where F_0 and F_1 denote the fault sets of AQ_{n-1}^0 and AQ_{n-1}^1 respectively. Similarly, F_X is the set of faulty cross edges while F_C denotes the set of faulty complement edges. We may assume that $|F_0| \geq |F_1|$. We now divide the proof into four cases:

Case 1: $|F_0| = 2n-1$. Then $|F_1 \cup F_C \cup F_X| = 0$. Note that $AQ_n - F$ has no isolated vertices, so we will show that it has either a perfect matching or an almost perfect matching. We may assume that $F = F_0$ contains vertices; otherwise, the result follows from Theorem 2.1. We pick two elements from F_0 to form A. We either pick two vertices or one vertex together with an edge so that $F_0 - A$ contains an even number of vertices. Let $F_0' = F_0 - A$. By construction, $AQ_{n-1}^0 - F_0'$ is an even graph. Suppose there is an isolated vertex v_1 in $AQ_{n-1}^0 - F_0'$. So every vertex in F_0' is adjacent to v_1 and every edge in F_0' is incident to v_1. Since F_0' has an even number of vertices and the degree of v_1 in AQ_{n-1}^0 is odd, F_0'

contains at least one edge say, (v_1, u_1). If A consists of a vertex and an edge e. Then let $A' = (A - \{e\}) \cup \{(v_1, u_1)\}$ and it is easy to see that $AQ^0_{n-1} - (F_0 - A')$ does not have an isolated vertex, and we may choose A' instead of A. Now suppose that A consists of two vertices y_1 and y_2. (Recall that F'_0 has an even number of vertices.) If F'_0 has a vertex z, then we may choose $A' = \{y_1, z\}$ and it is easy to see that $AQ^0_{n-1} - (F_0 - A')$ does not have an isolated vertex. Thus we assume that F'_0 consists of edges only. Then F has two vertices and $2n - 3$ edges. We claim that $AQ_n - F$ has a perfect matching. Now by the induction hypothesis, $AQ^0_{n-1} - \{y_1, y_2, v_1\} = AQ^0_{n-1} - (F_0 \cup \{v_1\})$ has an almost perfect matching M_0 missing, say, w. Consider the two cross edges (v_1, v_1^x) and (w, w^x). By the induction hypothesis, $AQ^1_{n-1} - \{v_1^x, w^x\}$ has a perfect matching M_1. Now $M_0 \cup M_1 \cup \{(v_1, v_1^x), (w, w^x)\}$ is a perfect matching in $AQ_n - F$, as required.

Henceforth, we may assume that $AQ^0_{n-1} - F'_0$ has no isolated vertices. Recall that by construction, $AQ^0_{n-1} - F'_0$ is an even graph. So by the induction hypothesis, $AQ^0_{n-1} - F'_0$ has a perfect matching M_P. We consider two subcases.

Subcase 1a: A contains distinct vertices v_1, v_2 in AQ^0_{n-1}. So F has an even number of vertices and we want to find a perfect matching in $AQ_n - F$. If v_1 and v_2 are adjacent then $(v_1, v_2) \in M_P$ and it is easy to extend it to a perfect matching in $AQ_n - F$. So we may assume that M_P matches v_1 and v_2 to the vertices v'_1 and v'_2, respectively in $AQ^0_{n-1} - F'_0$. Consider the cross edges (v'_1, v'^x_1) and (v'_2, v'^x_2). By the induction hypothesis, $AQ^1_{n-1} - \{v'^x_1, v'^x_2\}$ has a perfect matching M_1. Now, $(M_P - \{(v_1, v'_1)(v_2, v'_2)\}) \cup M_1 \cup \{(v'_1, v'^x_1), (v'_2, v'^x_2)\}$ is a perfect matching in $AQ_n - F$, as required.

Subcase 1b: A contains an edge (v, v') and a vertex u. (By assumption, $u \notin \{v, v'\}$ and $v, v' \notin F$.) So F has an odd number of vertices and we want to find an almost perfect matching in $AQ_n - F$. Now let $(u, u') \in M_P$. We consider whether the edge (v, v') is part of the matching M_P or not. If not, then $M_P - \{(u, u')\}$ is an almost perfect matching in $AQ^0_{n-1} - F_0$ missing u', which can be extended to an almost perfect matching in $AQ_n - F$ missing u', by using a perfect matching in $AQ^1_{n-1} = AQ^1_{n-1} - F_1$. Now assume instead that $(v, v') \in M_P$. So $M_P - \{(v, v'), (u, u')\}$ matches every vertex in $AQ^0_{n-1} - F_0$ except v, v', and u'. Since each vertex has a complement and cross edge incident with it, we simply choose the cross edges and match v and v' to the vertices v^x and v'^x in $AQ^1_{n-1} - F_1$. Since $|F_1| = 0$, it follows from the induction hypothesis that $AQ^1_{n-1} - \{v^x, v'^x\}$ has a perfect matching M_1. Furthermore, $(M_P - \{(v,v'),(u,u')\}) \cup M_1 \cup \{(v,v^x)(v',v'^x)\}$ is an almost perfect matching in $AQ_n - F$ missing u', so we are done.

Case 2: $|F_0| = 2n-2$. Then $|F_1 \cup F_C \cup F_X| = 1$. Note that $AQ_n - F$ has no isolated vertices, so we will show that it has either a perfect matching or an almost perfect matching.

Case 2a: F_0 contains only vertices. We consider two possibilities. The first possibility is that the unique element in $F_1 \cup F_C \cup F_X$ is an edge. Then let A be a set containing two elements of F_0, say u and v. By the induction hypothesis, $AQ^0_{n-1} - (F_0 - A)$ has a perfect matching M_0. If (u,v) is an edge and it is in M_0, then it is easy to extend M_0 to a perfect matching in $AQ_n - F$. So we may assume otherwise, and that $(u,u'),(v,v') \in M_0$. Since $F_1 \cup F_C \cup F_X$ contains exactly one edge, either $\{(u',u'^x),(v',v'^x)\} \cap F = \emptyset$ or $\{(u',u'^c),(v',v'^c)\} \cap F = \emptyset$. we may assume that (u',u'^x) and (v',v'^x) are not in F. Now by the induction hypothesis, $AQ^1_{n-1} - (F_1 \cup \{u'^x, v'^x\})$ has a perfect matching M_1. Then $(M_0 - \{(u,u'),(v,v')\}) \cup M_1 \cup \{(u',u'^x),(v',v'^x)\}$ is a perfect matching in $AQ_n - F$. The second possibility is the unique element in $F_1 \cup F_C \cup F_X$ is a vertex y in AQ^1_{n-1}. We consider the following scenrios.

- Suppose $y^c, y^x \in F_0$. Let $A = \{y^c, y^x\}$. By the induction hypothesis, $AQ^0_{n-1} - (F_0 - A)$ has a perfect matching M_0. If (y^c, y^x) is in M_0, then it is easy to extend M_0 to an almost perfect matching in $AQ_n - F$. So we may assume otherwise, and that $(y^c,u'),(y^x,v') \in M_0$. Clearly neither u' nor v' is adjacent to y. So we may assume that (u',u'^x) and (v',v'^x) are in $AQ_n - F$. Now by the induction hypothesis, $AQ^1_{n-1} - (F_1 \cup \{u'^x, v'^x\}) = AQ^1_{n-1} - (\{y\} \cup \{u'^x, v'^x\})$ has an almost perfect matching M_1. Then $(M_0 - \{(y^c,u'),(y^x,v')\}) \cup M_1 \cup \{(u',u'^x),(v',v'^x)\}$ is an almost perfect matching in $AQ_n - F$.
- Suppose exactly one of y^c and y^x is in F_0. Without loss of generality, we may assume that $y^c \in F_0$ and $y^x \notin F_0$. Let v be a vertex in F_0 that is neither y^c nor y^x. Let $A = \{y^c, v\}$. By the induction hypothesis, $AQ^0_{n-1} - (F_0 - A)$ has a perfect matching M_0. If (y^c, v) is an edge, and it is in M_0, then it is easy to extend M_0 to an almost perfect matching in $AQ_n - F$. So we may assume that $(y^c,u'),(v,v') \in M_0$. By

construction, at most one of u' and v' is adjacent to y. So we may apply the usual argument.

- Suppose $y^c, y^x \notin F_0$. Then let A be a set containing two elements of F_0, say u and v. By the induction hypothesis, $AQ^0_{n-1} - (F_0 - A)$ has a perfect matching M_0. If (u, v) is an edge, and it is in M_0, then it is easy to extend M_0 to an almost perfect matching in $AQ_n - F$. So we may assume otherwise, and that $(u, u'), (v, v') \in M_0$. If at most one of u' and v' is adjacent to y, we may apply the usual argument. Otherwise $\{u', v'\} = \{y^c, y^x\}$ and hence u' is adjacent to v'. Thus, $(M_0 - \{(u, u'), (v, v')\}) \cup \{(u', v')\}$ is a perfect matching in $AQ^0_{n-1} - F_0$ and it is easy to extend it to an almost perfect matching in $AQ_n - F$.

Case 2b: F_0 has at least one edge. If F_0 has an even number of vertices, then let A be a set containing an edge from F_0; otherwise let A be a set containing a vertex from F_0. Let $F'_0 = F_0 - A$. By construction, $AQ^0_{n-1} - F'_0$ is an even graph. Suppose there is an isolated vertex v_1 in $AQ^0_{n-1} - F'_0$. So every vertex in F'_0 is adjacent to v_1 and every edge in F'_0 is incident to v_1. Since F'_0 has an even number of vertices and the degree of v_1 in AQ^0_{n-1} is odd, F'_0 contains at least one edge say, (v_1, u_1). If A consists of an edge e, then let $A' = (A - \{e\}) \cup \{(v_1, u_1)\}$ and it is easy to see that $AQ^0_{n-1} - (F_0 - A')$ does not have isolated vertices, and we may choose A' instead of A. Now suppose that A consists of a vertex y. By construction, F'_0 has an even number of vertices. If F'_0 has a vertex z, then we may choose $A' = \{z\}$ and it is easy to see that $AQ^0_{n-1} - (F_0 - A')$ does not have any isolated vertices. Thus we assume that F'_0 consists of edges only. Then F_0 has one vertex, namely, y. By the induction hypothesis, $AQ^0_{n-1} - \{y, v_1\} = AQ^0_{n-1} - (F_0 \cup \{v_1\})$ has a perfect matching M_0. If the unique element in $F_1 \cup F_C \cup F_X$ is an edge, then F has one vertex, and we claim that $AQ_n - F$ has an almost perfect matching. Now M_0 together with a perfect matching in $AQ^1_{n-1} - F_1$ will be a desired almost perfect matching in $AQ_n - F$ missing v_1. Now suppose the unique element in $F_1 \cup F_C \cup F_X$ is a vertex (in AQ^1_{n-1}), say w_1. Then F has two vertices, and we claim that $AQ_n - F$ has a perfect matching. Clearly w_1 cannot be both v_1^x and v_1^c, so we may assume that it is not v_1^x. Let M_1 be a perfect matching in $AQ^1_{n-1} - \{v_1^x, w_1\}$ Then $M_0 \cup M_1 \cup \{(v_1, v_1^x)\}$ is a perfect matching in $AQ_n - F$, as required.

Henceforth, we may assume that $AQ^0_{n-1} - F'_0$ has no isolated vertices. Recall that by construction, $AQ^0_{n-1} - F'_0$ is an even graph. So by the induction hypothesis, $AQ^0_{n-1} - F'_0$ has a perfect matching M_P. We consider two subcases.

Subcase 2b(i): The element in A is the edge (v_1, v_2) in AQ^0_{n-1}. We note that if $(v_1, v_2) \notin M_P$, then it is easy to find a perfect matching or an almost perfect matching in $AQ_n - F$. So we may assume that $(v_1, v_2) \in M_P$. Since $|F_1 \cup F_C \cup F_X| = 1$, we claim that we can match v_1 and v_2 to vertices in the graph $AQ^1_{n-1} - F_1$. If our claim is correct, then we may assume that (v_1, v_1^c) and (v_2, v_2^c) are edges in $AQ_n - F$. Let M_1 be a perfect matching or an almost perfect matching in $AQ^1_{n-1} - (F_1 \cup \{v_1^c, v_2^c\})$ depending on whether F_1 contains a vertex. Then $(M_P - \{(v_1, v_2)\}) \cup M_1 \cup \{(v_1, v_1^c), (v_2, v_2^c)\}$ is the desired perfect matching or almost perfect matching in $AQ_n - F$. Now, our claim is clearly true if the unique element in $F_1 \cup F_C \cup F_X$ is an edge. Suppose the unique element in $F_1 \cup F_C \cup F_X$ is the vertex y in AQ^1_{n-1}. Then the claim is still true (by using either cross edges or complement edges) unless y is adjacent to both v_1 and v_2. In this case, we may assume that $y = v_1^x$ and $y = v_2^c$. Since the element in A is an edge, F_0 must contain an even number of vertices. Thus F_0 has an even number of edges and hence at least two edges. The natural argument is to choose another edge from F_0 to form A instead of using (v_1, v_2). However, we have already assumed that such an edge is chosen so that $AQ^0_{n-1} - (F_0 - A)$ has no isolated vertices. So care must be taken. If such an exchange produces an isolated vertex, then we may assume that the isolated vertex is v_1. If there is another edge (v_1, v_3) belonging to F_0, then we can use $A' = \{(v_1, v_3)\}$ instead. Now it is easy to see that $AQ^0_{n-1} - (F_0 - A)$ has no isolated vertices and v_3 is not adjacent to y. So we can match v_1 and v_3 to vertices in the graph $AQ^1_{n-1} - F_1$.

So if our claim is not correct, then F_0 consists of two edges, one of them is (v_1, v_2), and $2n-4$ neighbors of v_1, except v_2. Moreover, the unique element in $F_1 \cup F_C \cup F_X$ is the vertex y in AQ^1_{n-1}, and y is adjacent to v_1 and v_2. We may assume that $v_1^c \notin F$. (So $y = v_1^x$.) We will use a different construction. Let $w, z \in F_0$, both neighbours of v. Let $F''_0 = F_0 - \{w, z\}$. By the induction hypothesis, $AQ^0_{n-1} - F''_0$ has a perfect matching M_0. So we may assume that $(v_1, w), (z, z') \in M_0$. By the induction hypothesis, there is a perfect matching M_1 in $AQ^1_{n-1} - \{y, v^c\}$. Now $(M_0 - \{(v_1, w), (z, z')\}) \cup M_1 \cup \{(v_1, v_1^c)\}$ is an almost perfect matching in $AQ_n - F$ missing z'.

Subcase 2b (ii): The element in A is the vertex v_1 in AQ_{n-1}^0. Let $(v_1, v_2) \in M_P$. Since $|F_1 \cup F_C \cup F_X| = 1$ and we may use either cross edges or complementary edges, we may assume that (v_2, v_2^c) is in $AQ_n - F$. Let M_1 be a perfect matching or an almost perfect matching in $AQ_{n-1}^1 - (F_1 \cup \{v_2^c\})$ depending on whether F_1 contains a vertex. Then $(M_P - \{(v_1, v_2)\}) \cup M_1 \cup \{(v_2, v_2^c)\}$ is the desired perfect matching or almost perfect matching in $AQ_n - F$.

Case 3: $|F_0| = 2n - 3$. First, assume that F_0 is not a trivial strong matching preclusion set of $AQ_{n-1}^0 - F_0$. Then, by the induction hypothesis, each of $AQ_{n-1}^0 - F_0$ and $AQ_{n-1}^1 - F_1$ contains a perfect matching or an almost perfect matching. If at least one of $AQ_{n-1}^0 - F_0$ or $AQ_{n-1}^1 - F_1$ contains a perfect matching, we are done. So, we assume that both $AQ_{n-1}^0 - F_0$ and $AQ_{n-1}^1 - F_1$ have almost perfect matchings. Thus F_1 contains only one vertex. (It may contain another edge.) In particular, M_0 is an almost perfect matching of $AQ_{n-1}^0 - F_0$ missing w. If w is isolated in $AQ_{n-1}^0 - F_0$, then either (w, w^x) or (w, w^c) is in $AQ_n - F$; otherwise w is isolated in $AQ_n - F$ and F is a basic optimal strong matching preclusion set of AQ_n, which implies F is a trivial optimal strong matching preclusion set of AQ_n by Proposition 1.2. So, for convenience, we assume that (w, w^c) is in $AQ_n - F$. Let M_1 be a perfect matching in $AQ_{n-1}^1 - (F_1 \cup \{w^c\})$. Then $M_0 \cup M_1 \cup \{(w, w^c)\}$ is a perfect matching in $AQ_n - F$. Therefore, we may assume that w is not isolated in $AQ_{n-1}^0 - F_0$. This implies $AQ_{n-1}^0 - F_0$ has no isolated vertices. Suppose $|\{w^c, w^x, (w, w^c), (w, w^x)\} \cap F| \leq 1$. Then we may assume that $w^c, (w, w^c) \notin F$. Thus $AQ_{n-1}^1 - (F_1 \cup \{w^c\})$ contains a perfect matching M_1, and hence $M_0 \cup M_1 \cup \{(w, w^c)\}$ is a perfect matching in $AQ_n - F$. So we may assume that $|\{w^c, w^x, (w, w^c), (w, w^x)\} \cap F| = 2$ and neither (w, w^c) nor (w, w^x) are in $AQ_n - F$. So, without loss of generality, we may assume that $w^c, (w, w^x) \in F_1$. We apply Proposition 3.3 to find another almost perfect matching in $AQ_{n-1}^0 - F_0$ missing $y \neq w$. Now $|\{y^c, y^x, (y, y^c), (y, y^x)\} \cap F| \leq 1$ and we can repeat the argument.

Henceforth, we may assume that F_0 is a trivial strong matching preclusion set of AQ_{n-1}^0. So there exists an isolated vertex v in $AQ_{n-1}^0 - F_0$ and F_0 contains vertices that are adjacent to v or edges that are incident to v. Moreover, F_0 contains an even number of vertices. If neither (v, v^c) nor (v, v^x) is in $AQ_n - F$, then F is basic in AQ_n and hence trivial in $AQ_n - F$ by Proposition 1.2. So, for convenience, we may assume that (v, v^c) is in $AQ_n - F$. Let

f_1 and f_2 be two elements of F_0 and let $F_0' = F_0 - \{f_1, f_2\}$. Clearly we can pick f_1 and f_2 to either both be vertices or both be edges. By Theorem 2.3, $AQ^0_{n-1} - F_0'$ contains a Hamiltonian cycle C since $|F_0'| = 2n-5$. If f_1 and f_2 are both edges, then both are incident to v and they are in C; thus $C - \{v\}$ is a path P in $AQ^0_{n-1} - F_0$ with an odd number of vertices. If f_1 and f_2 are both vertices, then both are adjacent to v and they are on C; thus $C - \{v, f_1, f_2\}$ is a path P in $AQ^0_{n-1} - F_0$ with an odd number of vertices. Now P contains at least $2^{n-1} - (2n-3) - 1 = 2^{n-1} - 2n + 2$ vertices. For at least $(2^{n-1} - (2n-3) - 1)/2 = 2^{n-2} - n + 1$ vertices, its deletion will separate P into two paths, each with an even number of vertices. Since $2^{n-2} - n + 1 \geq 4$ as $n \geq 5$, there is one such vertex z such that (z, z^c) is in $AQ_n - F$. Thus C induces a matching M_0 in $AQ^0_{n-1} - F_0$ missing z. Now by the induction hypothesis, $AQ^1_{n-1} - (F_1 \cup \{v^c, z^c\})$ has a perfect matching or an almost perfect matching M_1. Then $M_0 \cup M_1 \cup \{(v, v^c), (z, z^c)\}$ is either a perfect matching or an almost perfect matching in $AQ_n - F$.

Case 4: $|F_0| < 2n-3$. By the induction hypothesis, $AQ^0_{n-1} - F_0$ has a perfect or an almost perfect matching. Moreover, $|F_1 \cup F_X \cup F_C| > 2$. Since $|F_0| \geq |F_1|$, $AQ_{n-1} - F_1$ also contains a perfect or an almost perfect matching. If at least one of $AQ^0_{n-1} - F_0$ and $AQ^1_{n-1} - F_1$ has a perfect matching, we are done. So, assume both $AQ^0_{n-1} - F_0$ and $AQ^1_{n-1} - F_1$ have an odd number of vertices. We consider two subcases.

Subcase 4a: $|F_0| \leq 2n-5$. Since there are 2^{n-1} cross edges and 2^{n-1} complement edges, we may assume that there is at least one cross edge and one complement edge not on F as $2^{n-1} > 2n-1$ for all $n \geq 5$. We consider such a fault-free complement edge (v, v^c) between $AQ^0_{n-1} - F_0$ and $AQ^1_{n-1} - F_1$ where v is in $AQ^0_{n-1} - F_0$. Note that by assumption, both $AQ^0_{n-1} - (F_0 \cup \{v\})$ and $AQ^1_{n-1} - (F_1 \cup \{v^c\})$ contain an even number of vertices. Now, $|F_0 \cup \{v\}|, |F_1 \cup \{v^c\}| \leq 2n-4$. So $AQ^0_{n-1} - (F_0 \cup \{v\})$ and $AQ^1_{n-1} - (F_1 \cup \{v^c\})$ have perfect matchings M_0 and M_1, respectively. Thus $M_0 \cup M_1 \cup \{(v, v^c)\}$ is a perfect matching in $AQ_n - F$.

Subcase 4b: $|F_0| = 2n-4$. Hence $|F_1 \cup F_X \cup F_C| = 3$. We note that F_1 contains either one vertex or three vertices. Thus $|F_X \cup F_C| \leq 2$. We start with an almost perfect matching M_0 missing w in $AQ^0_{n-1} - F_0$. Suppose $|\{w^c, w^x, (w, w^c), (w, w^x)\} \cap F| \leq 1$. Then we may assume that $w^c, (w, w^c) \notin F$. Thus $AQ^1_{n-1} - (F_1 \cup \{w^c\})$ contains a perfect matching

M_1, and hence $M_0 \cup M_1 \cup \{(w, w^c)\}$ is a perfect matching in $AQ_n - F$. (We note that $|F_1 \cup \{w^c\}| \leq 4 \leq< 2n-3$ as $n \geq 5$.) So we may assume that $|\{w^c, w^x, (w, w^c), (w, w^x)\} \cap F| \geq 2$. Thus the construction will not work. One may want to apply Proposition 3.3 to find another almost perfect matching in $AQ^0_{n-1} - F_0$ missing $y \neq w$. However, it is possible that $|\{y^c, y^x, (y, y^c), (y, y^x)\} \cap F| \geq 2$. (To be precise, this happens when $w^c = y^c$ and $w^x = y^c$.) Instead, we apply Proposition 3.4. Clearly $AQ^0_{n-1} - F_0$ has no isolated vertices. If there is a forbidden 2-path $w - u - v$ in $AQ^0_{n-1} - F_0$ where both v and w are of degree 1, then we can completely determine F_0. Since $|F_0| = 2n - 4$ and F_0 contains an odd number of vertices, F_0 must contain exactly one edge (w, v) and $2n - 5$ vertices, each adjacent to both w and v. But such a configuration is impossible in AQ^0_{n-1}. (Otherwise, deleting these $2n - 5$ vertices together with u will disconnect the graph, which is impossible as AQ^0_{n-1} has connectivity $2n - 3$ since $n \geq 5$.) Since we have three different almost perfect matchings in $AQ^0_{n-1} - F_0$, each missing a different vertex, we may indeed assume there is an almost perfect matching missing w in $AQ^0_{n-1} - F_0$ such that $|\{w^c, w^x, (w, w^c), (w, w^x)\} \cap F| \leq 1$, so we are done. $\square$

4. Conclusion

In this paper, we studied the strong matching preclusion problem introduced in [25]. Given hypercubes are bipartite and hence not resilient under the strong matching preclusion measure as shown in [25], it is natural to consider non-bipartite variants of hypercubes. The class of augmented cubes is a natural choice due to its many attractive properties. We showed that these interconnection networks are indeed resilient under this measure.

Acknowledgements:
We would like to thank the three anonymous referees for a number of helpful comments and suggestions.

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