A few strong knapsack facets*

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Abstract

We perform a shooting experiment for the knapsack facets and observe that $1/k$-facets are strong for small k; in particular, k dividing 6 or 8. We also observe spikes of the size of $1/k$-facets when $k = n$ or when $k + 1$ divides $n + 1$. We discuss the strength of the $1/n$-facets introduced by Aráoz et al. [2] and the knapsack facets given by Gomory's homomorphic lifting.

A general integer knapsack problem is a knapsack subproblem where a portion, often a significant majority, of the variables are missing from the master knapsack problem. The number of projections of $1/k$-facets on a knapsack subproblem of l variables is $O(l^{\lceil k/2 \rceil})$, note that this is independent of the size of the master problem. Since $1/k$-facets are strong for small k, we define the $1/k$-inequalities which include the $1/d$-facets with d dividing k and fix k to be a small constant such as $k = 6$ or $k = 8$. We develop an efficient way of enumerating violated valid $1/k$-inequalities. For each violated $1/k$-inequality, we determine its validity by solving a small integer programming problem, the size of which depends only on k.

1 Introduction

The *master knapsack problem* of order n is defined to be

$$\max \quad vt \tag{1}$$

$$st \quad \sum_{i=1}^{n} i t_i = n \tag{2}$$

$$t \geq 0 \tag{3}$$

$$t_i \text{ are integers,} \tag{4}$$

where $v \geq 0$ is a row vector of length n, and t is a column vector of n variables. Observe that equation (2) contains all integer coefficients from 1 to n. The problem given by (1)-(4) is known as the *master knapsack problem*, $K(n)$.

The convex hull of the solutions to $K(n)$ is denoted by $P(K(n))$ and referred to as the *master knapsack polytope*. The dimension of $P(K(n))$ is $n - 1$ and the non-negativity constraints (3) are facet-defining for $i \geq 2$ (see Shim [16] and Shim and Johnson [18].) We call the facet-defining non-negativity constraints *trivial* facets. The other facets are called *knapsack facets*. Since $P(K(n))$

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is not full dimensional, each knapsack facet has infinitely many representations. Throughout this paper, we consider the representation $\xi t \leq 1$ with $\xi_1 = 0$ and $\xi_n = 1$, where ξ is the length n row vector of coefficients. Overloading our notation we use ξ as a representative of $\xi t \leq 1$ and refer to this vector itself as a knapsack facet. A knapsack facet $\xi t \leq 1$ is called a *$1/k$-facet* if k is the smallest possible integer such that

$$\xi_i \in \{0/k, 1/k, 2/k, ..., k/k\} \cup \{1/2\}. \tag{5}$$

Note that k is the least common multiple of the denominators of the irreducible fractions ξ_i except $\xi_i = 1/2$. The set of indices i where the coefficients $\xi_i = 1/2$ is referred to as the *half landing.* The half-landing of a knapsack facet is known in [17] to have indices i with $n/3 < i < 2n/3$. We call a $1/k$-facet *strict* if m/k appears as a coefficient for every $m \in \{1, 2, ..., k\} \setminus \{k/2\}$.

The remainder of the paper is organized as follows. In Section 2, we introduce some specific classes of knapsack facets to be further discussed and studied. In Section 3, we describe the results of a shooting experiment and observe that $1/k$-facets are important for k dividing 6 or 8. We also observe importance of $1/k$-facets when $k = n$ or when $k + 1$ divides $n + 1$.

Knapsack subproblems are restrictions of the master knapsack problem where the variables t_i only appear for $i \in L$ where $L \subseteq \{1, \ldots, n\}$, i.e., some of the coefficients do not appear in the equation. The convex hull of the feasible solutions to a knapsack subproblem is given by

$$P(K(n)) \cap \{t : t_i = 0 \ \ \forall i \notin L\}.$$

Knapsack subproblems provide a connection between the master knapsack problem and many single row relaxations of general integer programs. The projections of the knapsack facets are valid inequalities for a knapsack subproblem.

In Section 4, we develop an algorithms to separate and enumerate $1/k$-facets for some small values of k for knapsack subproblems. The number of projections of $1/k$-facets for a knapsack subproblem of l variables is $O(l^{\lceil k/2 \rceil})$, which is independent of the size of the master problem. We therefore are focused on the use of separation and enumeration algorithms for the knapsack subproblems whose running time are independent of n.

In Section 5, we discuss a notion of the strength of a knapsack facet and analyze the strength of a class of $1/n$-facets, which were observed to be important in the shooting experiment. In Section 6, we discuss some special classes of knapsack subproblems where $1/k$-facets with small k are likely to be ineffective and suggest alternative solution methods, such as group relaxations. Finally, Section 7 gives concluding remarks.

2 Characterization of $1/k$-facets

Aráoz [1] characterized the knapsack facets as the extreme rays of a polynomially sized system of super-additive relations. Hunsaker [14] described the knapsack facets by the extreme points of the system with fixing $\xi_1 = 0$.

Theorem 2.1 (Aráoz [1], Hunsaker [14]) *The coefficient vectors ξ of the knapsack facets $\xi t \leq$*

1 of $K(n)$ with $\xi_1 = 0$ and $\xi_n = 1$ are the extreme points of the system of linear constraints

$$\xi_1 = 0, \tag{6}$$

$$\xi_n = 1, \tag{7}$$

$$\xi_i + \xi_{n-i} = 1 \qquad \text{for } 1 \le i \le n/2, \tag{8}$$

$$\xi_i + \xi_j \le \xi_{i+j} \qquad \text{whenever } i + j < n. \tag{9}$$

The feasible solutions to the system give valid inequalities $\xi t \le 1$ for $P(K(n))$.

We call (8) and (9) the *complementarities* and the *superadditivities*, respectively. Therefore, a knapsack facet ξ is a non-decreasing sequence because

$$\xi_i = 0 + \xi_i = \xi_1 + \xi_i \le \xi_{i+1}.$$

Although $P(K(n))$ has exponentially many facets, certain polynomially sized subsets of these facets appear to be of special importance. In the following we review some of these classes.

2.1 $1/k$-inequalities

In this paper, a sequence $\xi = (\xi_i)_{i=1}^n$ is called *symmetric* if the complementarities (8) hold. We call $\xi t \le 1$ a $1/k$-*inequality* if ξ is a non-decreasing symmetric sequence that satisfies (5). In general, a $1/k$-inequality need not be a valid inequality for $P(K(n))$. We see that a $1/d$-inequality is a $1/k$-inequality if d is a divisor of k. A $1/k$-inequality ξ is uniquely determined by a non-decreasing sequence (a_m) where a_m represents the first index i with $\xi_i \ge m/k$ for $m \in \{0, 1, ..., k\} \cup \{k/2\}$. Observe that $k/2$ is not an integer for k odd but is required to obtain the coefficient $1/2$. If $a_m = a_{m+1}$ or if $a_m = a_{m+1/2}$ with k odd and $m \in \{(k-1)/2, k/2\}$, no coefficient ξ_i has value m/k. Such a sequence ξ will be denoted by $\xi^{k\text{-}(a_m)}$. Also, because of symmetry, the number of ξ_i's of value m/k must equal the number of those of value $(k-m)/k$. Thus, a_m for $m = 1, ..., k/2$ are sufficient to uniquely define $\xi^{k\text{-}(a_m)}$. A $1/k$-inequality is called a $1/k$-*facet* if it is a knapsack facet and if k is the smallest possible integer that satisfies (5). We remark that every facet is a $1/k$-facet for some value of k. However, fixing k allows us to consider specific, polynomially sized, classes of facets; we are particularly interested in certain small fixed values of k, the importance of which have been demonstrated by Shim, Cao and Chopra [17].

2.2 $1/n$-facets

Aráoz et al. [2] defined two families of knapsack facets which are equivalent to $1/n$-facets. One family is defined in Theorem 6.5 of [2] and equivalent to the $1/n$-facets $\xi^{n\text{-}(a_m)}$ given by $a_1 = ... = a_q = q$ and by $a_i = \lceil i \rceil$ for $q \le i \le n - q$ where $1 < q \le \frac{n}{4}$.

The other family is defined in Theorem 6.3 of [2] and equivalent to the $1/n$-facets $\xi^{n\text{-}(a_m)}$ given by $a_i = i$ for $i < q$ even and $a_i = i + 1$ for $i < q$ odd and by $a_i = i$ for $q \le i \le n - q$ where $q \le \frac{n}{2}$ if n is even and $q \le \frac{n-2}{3}$ if n is odd. Any knapsack facet ξ with $\xi_2 = \frac{2}{n}$ belongs to the family, as shown in Aráoz et al. [2].

In Section 5.3, we analyze the strength of these two families of $1/n$-facets and see that the first family appears to be important, and the second one appears to be less important, this agrees with the findings in the shooting experiment described in Section 3.

2.3 Group homomorphically lifted facets

A knapsack problem may be relaxed to the cyclic group problem, defined by Gomory [8]. The *cyclic group problem* (C_n, b) with respect to a cyclic group C_n of order n and a non-zero element b, has a feasible region given by vectors t that satisfy

$$\sum_{g \in C_n \setminus \{0\}} g t_g = b,$$

where t_g, $g \in C_n \setminus \{0\}$, are non-negative integer variables. The convex hull of the integer solutions t of the problem was shown by Gomory [8] to be a polyhedron and is referred to as the *cyclic group polyhedron*, denoted $P(C_n, b)$. The non-negativity constraints $t_g \geq 0$, $g \neq 0$, are facets of $P(C_n, b)$ and called *trivial* facets. The non-trivial facets are denoted by $\pi t \geq \pi_b > 0$ and called *cyclic group facets*.

A *homomorphism* ϕ of a group G into a group H is a map $\phi : G \to H$ which preserves the addition; i.e.,

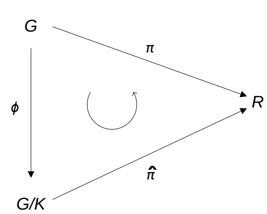
$$\phi(g_1 + g_2) = \phi(g_1) + \phi(g_2) \text{ for all } g_1, g_2 \in G.$$

The *kernel* $Ker(\phi)$ of the map $\phi : G \to H$ is defined to be the set of elements g in G which are mapped to $\phi(g) = 0$.

Gomory [8] showed a facet of a group polyhedron with a 0 coefficient can be constructed by repeating a facet of a lower dimensional group polyhedron. His lifting theorem enables us to assemble facets as building blocks for higher dimensional facets:

Theorem 2.2 *Let G be an abelian group and let K be a subgroup of G. Let $b \in G \setminus K$ and let $\phi : G \to G/K$ be the canonical homomorphism of G onto the factor group G/K. If $\hat{\pi}$ is a group facet for G/K with right-hand side $\hat{b} = \phi(b)$, then a group facet for G with right-hand side b is given by*

$$\pi(g) = \hat{\pi}(\phi(g)).$$



Lifted facets are shown to be important for the cyclic group polyhedron in the shooting experiment performed by Gomory, Johnson and Evans [12].

We refer to the vector $(1/n, 2/n, ..., n/n)$ as *lineality* and denote it by $lin(n)$, note that this is simply the coefficient vector of the knapsack equation divided by n. There is a connection between the knapsack facets and the facets of the cyclic group polyhedron; namely the knapsack facets for $P(K_n)$ are precisely the cyclic group facets which are adjacent to the lineality $lin(n)$ in $P(C_{n+1}, n)$ [2]. The following theorem states a tilting by which our representation $\xi t \leq 1$ of a knapsack facet can be converted into a cyclic group facet $\pi^{\xi} t \geq 1$ of $P(C_{n+1}, n)$.

Theorem 2.3 *Let* $sep(\xi) = \max_{i+j>n+1} \xi_i + \xi_j - \xi_{i+j-n-1}$ *and let*

$$\pi^{\xi} = \frac{n+1}{n \cdot sep(\xi) - (n+1)} \cdot (lin(n) - \xi) + lin(n).$$

Then, the knapsack facet $\xi t \leq 1$ *with* $\xi_1 = 0$ *and* $\xi_n = 1$ *can be alternatively represented by* $\pi^{\xi} t \geq 1$ *which is a cyclic group facet for* $P(C_{n+1}, n)$.

Since lifted cyclic group facets are important, we suspect the knapsack facets equivalent to lifted cyclic group facets to also be important. If $k+1$ divides $n+1$, there is a $1/k$-facet such that its equivalent cyclic group facet has a zero coefficient $\pi_i = 0$ and is a lifted facet; in particular, the repetition π^{ξ} of $lin\left(\frac{n+1}{k+1}\right)$ is a cyclic group facet of (C_{n+1}, n) having a zero coefficient. In Section 3, we observe that $1/k$-facets are strong when $k+1$ divides $n+1$.

3 Shooting experiment

Intuitively, the shooting experiment shoots an arrow from the origin toward the knapsack facets in a random direction sampled from the spherically uniform distribution in the non-negative orthant, and sees which facet is hit first. A random vector v follows the spherically uniform distribution in the non-negative orthant if v_i are the absolute values of the independent and identically normally distributed random variables with mean 0. While this conceptual process might seems to require checking exponentially many knapsack facets against the random direction, the facet hit by each shot can be determined in polynomial time by solving a single linear program. As done by Hunsaker [14], we may perform a shooting experiment by solving the *shooting linear programming problem* to maximize a random direction $v \geq 0$ over the constraints (6)-(9) with variables ξ. Its optimal solution ξ is the coefficient vector of the knapsack facet $\xi t \leq 1$ hit by shooting in v.

Shooting experiments were first used in the context of the TSP by Kuhn [15], and then by Gomory, Johnson and Evans [12], Evans [6] and Dash and Günlük [5] for the master cyclic group polyhedron.

3.1 Minimal characterization of the knapsack facets

We identify a minimal representation of the system (6)-(9) transforming the minimal representation in Shim [16].

Theorem 3.1 *A minimal representation of the system (6)-(9) is the equalities in (6)-(8) and the inequalities (9) replaced by*

$$\xi_i + \xi_j \leq \xi_{i+j} \text{ for } i \leq j < i+j < n/2, \tag{10}$$

$$\xi_i + \xi_j + \xi_{n-i-j} \leq 1 \text{ for } i \leq j \leq n-i-j < n/2, \tag{11}$$

$$\text{and } 2\xi\left(\frac{n}{4}\right) \leq \xi\left(\frac{n}{2}\right) = \frac{1}{2} \text{ if } n \equiv 0 \mod 4. \tag{12}$$

The shooting experiment is also equivalent to solving the linear programming problem to maximize

$$\sum_{1<i<n/2} (v_i - v_{n-i})\xi_i$$

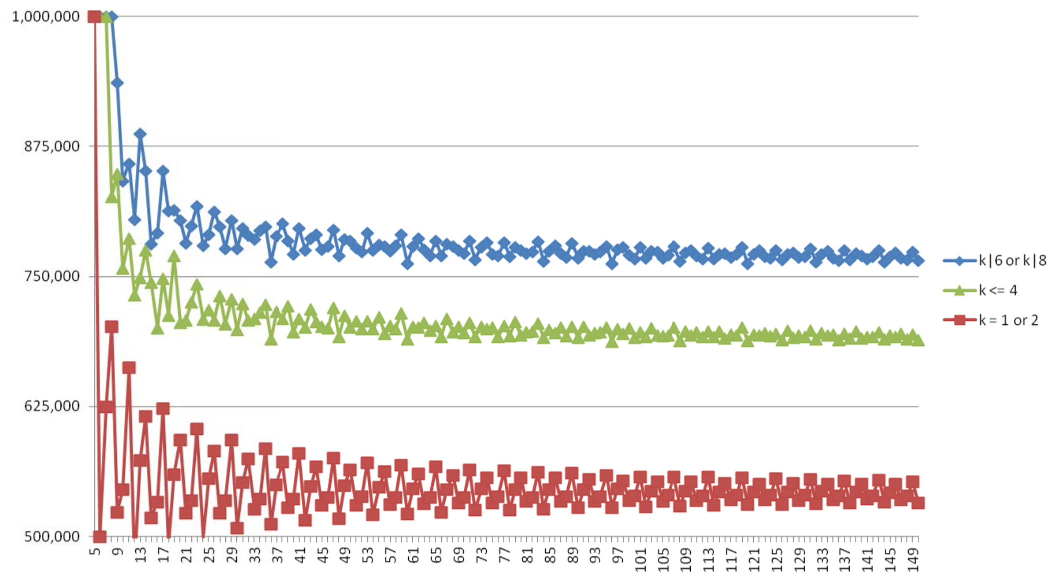


Figure 1: The number of hits absorbed by $1/k$-facets of $K(n)$ out of one million shots for each $n \leq 150$

over the system (10)-(12). Its optimal solution $(\xi_i : 1 < i < n/2)$ can be lifted by complementarities (8) to the coefficient vector $\xi = (\xi_i : i = 1, ..., n)$ of a knapsack facet $\xi t \leq 1$. The number of variables of the reduced shooting LP is half that of the original shooting LP and the number of constraints reduces to one third. Hence, Theorem 3.1 allows performing the shooting experiment for a larger n.

3.2 Concentration on $1/k$-facets with k dividing 6 or 8

Shim and Johnson [18] performed a shooting experiment firing off 10,000 shots for small order $n \leq 20$ and identified a pattern of the most hit knapsack facets. In every shooting experiment, the most hit facet was always the 1-facet of rank 0 and all the 1-facets absorbed more than 50% of hits except $n = 6, 12, 18$. This numerical experiment suggested that, if we look at the knapsack facets from the origin, the 1-facets, a family of linear size in n, will dominate our field of view, despite the fact that exponentially many facets are needed to describe $P(K(n))$. The 1-facets are shown in [18] to be adjacent to each other.

Figure 1 is a result of our new shooting experiment firing off one million shots for large order $n \leq 150$ and confirms that the 1-facets absorb more than 50% of the hits except for the cases $n = 6, 12, 18, 24$. The horizontal axis of the figure indicates $n = 5, ..., 150$ and the vertical axis indicates the number of hits absorbed by the 1-facets out of a million shots for each n. In the figure we see that the $1/k$-facets with k dividing 6 or 8 absorb more than 75% of shots.

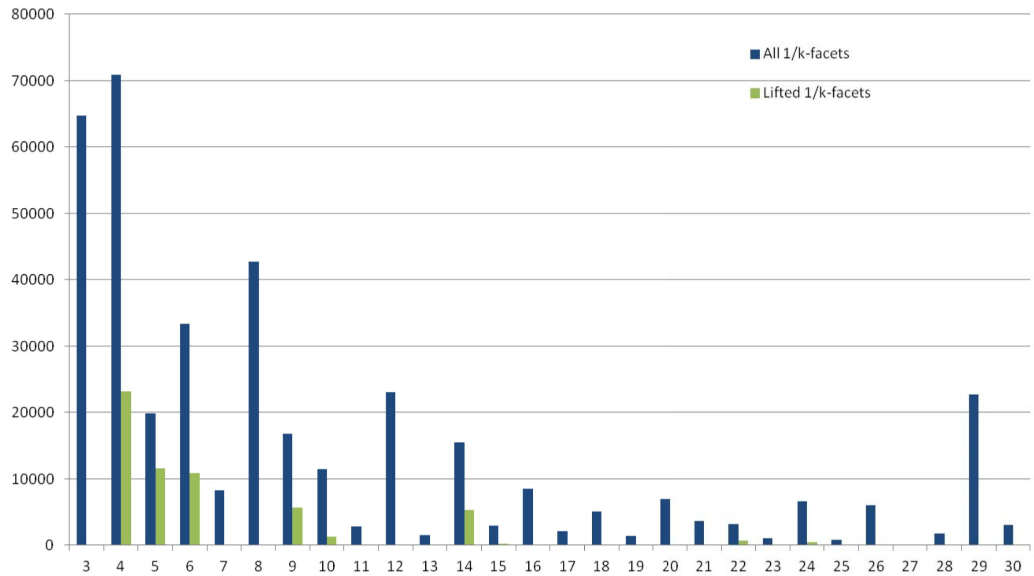


Figure 2: The number of hits absorbed by $1/k$-facets, and lifted $1/k$-facets of $K(29)$ out of one million shots. (Case $k = 2$ excluded from plot).

3.3 Decrease and oscillation of the size of $1/k$-facets

Figure 2 depicts the result of the shooting experiment for $K(29)$, the graph plots the number of shots absorbed by $1/k$-facets and lifted $1/k$-facets for each value of k up to 30. Here we see that the number of $1/k$-facets hit decreases as k grows and oscillates; i.e., $1/k$-facets are hit more often with k even than those with k odd. The worst case analysis of [17] suggests that the size of regular $1/k$-facets decreases and oscillates in the same manner. The result of shooting experiment goes together with the worst case analysis and we provide theoretical support for the results of the shooting experiment.

3.4 Spikes at $1/n$-facets and lifted facets

In Figure 2, we see two kinds of spikes; one at $k = n$ and the other at k with $k + 1$ dividing $n + 1$. The $1/n$-facets are frequently hit when n is small, and somewhat less hit as n grows. We are more interested in $1/k$-facets with small k. However, lifted facets are frequently hit for all values of n; in particular $1/k$-facets that correspond to repetitions of $lin(d)$ with $d = (n + 1)/(k + 1)$ when tilted to the π notation of (C_{n+1}, n) are frequently hit.

4 Separation and enumeration for sub-problems

Recall that a knapsack subproblem is a restriction of the master knapsack problem that is missing some terms in the knapsack equation (2); *i.e.*,

$$\sum_{i \in L} i t_i = n, \tag{13}$$

where L is a non-empty subset of $\{1, ..., n\}$. We are especially interested in cases when the dimension $l = |L|$ of the subproblem is much smaller than the size n of the master problem. The knapsack facets are valid inequalities for a knapsack subproblem.

4.1 Separation of the projections of 1-facets

Although the master knapsack problem has $\Theta(n)$ 1-facets, a knapsack sub-problem has only $O(l)$ unique projected 1-facets. We now develop a separation algorithm that will identify the 1-facet most violated by a given nonnegative solution $\hat{t}$ in time linear in l. Recall that a 1-facet $\xi^{2\text{-}(a_1,a_2)}$ is defined uniquely by the number a_1 which is known to satisfy $n/3 < a_1 \leq (n+1)/2$.

If the following quantity is greater than zero, it gives the amount by which a nonnegative solution $\hat{t}$ violates the 1-facet $\xi^{2\text{-}(a_1,a_2)}$ (if non-positive, it is the slack by which $\hat{t}$ satisfies the 1-facet):

$$VIOL(a_1) = \xi^{2\text{-}(a_1,a_2)}\hat{t} - 1 = \left(\sum_{i \in L, a_1 \leq i < n-a_1} \hat{t}_i/2 + \sum_{i \in L, i \geq n-a_1} \hat{t}_i \right) - 1. \tag{14}$$

To find the most violated 1-facet we must determine

$$\max_{n/3 < a_1 \leq (n+1)/2} VIOL(a_1). \tag{15}$$

However, in order to find a_1 maximizing the quantity in formula (15) we observe that many values of a_1 need not be considered because L does not contain all indices between 1 and n. Namely, it is sufficient to consider $a_1 \in X$ where $X = \{x : n/3 < x \leq (n+1)/2$ and either $x \in L$ or $n - x \in L\}$. Essentially, X gives a set of indices that are feasible for a_1 and also correspond to indices where L has a nonzero component in position a_1, or the complementary position $a_2 = n - a_1$. (There is one exceptional case when X is empty, in such case it is enough to consider the 1-facet given by $a_1 = \lfloor (n+1)/2 \rfloor$, which can be easily checked.)

We assume that the values in $X = \{x_1, \ldots, x_m\}$ are in sorted order. Note that $VIOL(x_1)$ may be computed in $O(l)$ time using formula (14). Additionally, if $1 < i \leq m$ then

$$VIOL(x_i) = VIOL(x_{i-1}) + \frac{1}{2}(\hat{t}_{x_i} - \hat{t}_{n-x_i}) \tag{16}$$

where $\hat{t}_k$ is understood to be zero whenever $k \notin L$. Noting that $|X| = O(l)$ we see that this naturally leads to a dynamic programming style $O(l)$ time algorithm to evaluate formula (15) and find the 1-facet most violated by $\hat{t}$. A simple modification to this algorithm would also allow us to enumerate all violated inequalities from this class.

4.2 Enumeration of the violated projections of $1/k$-inequalities

The number of the projections of the $1/k$-inequalities is $O\left(l^{\lceil k/2 \rceil}\right)$ which is independent of n, the size of the master knapsack problem $K(n)$. If we enumerate the $1/k$-inequalities one by one and compute $LHS = \xi_L t_L$ in linear time $O(l)$ to see violation or $LHS > 1$ for each $1/k$-inequality, the total time for enumerating the violated $1/k$-inequalities is $O\left(l^{\lceil k/2 \rceil + 1}\right)$. In a similar manner to Section 4.1, we can enumerate the violated projections of $1/k$-inequalities keeping time $O\left(l^{\lceil k/2 \rceil}\right)$ by updating $\xi_L t_L$ from iteration to iteration (in constant time) instead of computing it from scratch for each ξ_L. However, as noted previously not all $1/k$-inequalities are valid; in the next section we describe a very small integer program that can be used to determine if ξ_L defines a valid inequality.

4.3 Validity of the projections of $1/k$-inequalities

For each violated projection ξ_L of a $1/k$-inequality, we must check if there is a coefficient vector ξ of a valid inequality satisfying (6)-(9), implying the validity of ξ_L. The following gives a polyhedral description of the necessary relationships between the elements of the sequence (a_m), defining the coefficients of a $1/k$-inequality $\xi^{k\text{-}(a_m)}$.

Lemma 4.1 *A $1/k$-inequality $\xi^{k\text{-}(a_m)}$ satisfies (6)-(9) if and only if*

$$2 \leq a_{m_1} \leq a_{m_2} \leq (n+1)/2 \qquad \text{for } m_1 \leq m_2, \tag{17}$$

$$a_m + a_{k+1-\lceil m \rceil} = n+1 \qquad \text{for } m \leq k/2, \tag{18}$$

$$a_{m_1} + a_{m_2} \geq a_{\lceil m_1 + m_2 \rceil} \qquad \text{for all } m_1 \leq m_2 \text{ with } \lceil m_1 + m_2 \rceil \leq k. \tag{19}$$

In order to check if there is ξ satisfying (6)-(9), we can check if there is an integer solution (a_m) satisfying (17)-(19). Note that the system has $\lceil k/2 \rceil$ variables and $O(k^2)$ constraints. The number of variables and the number of constraints are independent of l and constant if k is fixed to be a constant. We now give two examples illustrating how this technique can be applied.

Example 4.2 A 1/6-inequality $\xi^{6\text{-}(a_m)}$ is valid if there is an integer solution (a_m) satisfying

1. Complementarities: $a_1 + a_6 = n+1$, $a_2 + a_5 = n+1$, $a_3 + a_4 = n+1$
2. Non-decreasing: $2 \leq a_1 \leq a_2 \leq a_3 \leq (n+1)/2$
3. Sub-additivities: $2a_1 \geq a_2$, $a_1 + a_2 \geq a_3$; $a_1 + 2a_3 \geq n+1$, $2a_2 + a_3 \geq n+1$
4. Defining: $x_{i_1 - 1} + 1 \leq a_1 \leq x_{i_1}$, $x_{i_2 - 1} + 1 \leq a_2 \leq x_{i_2}$, $x_{i_3 - 1} + 1 \leq a_3 \leq x_{i_3}$

where $x_{i_1}, x_{i_2}, x_{i_3} \in X$ are the smallest indices $x \in X$ of $\xi_x \geq 1/6, 2/6, 3/6$.

Example 4.3 A 1/8-inequality $\xi^{8\text{-}(a_m)}$ is valid if there is an integer solution (a_m) satisfying

1. Complementarities: $a_1 + a_8 = n+1$, $a_2 + a_7 = n+1$, $a_3 + a_6 = n+1$, $a_4 + a_5 = n+1$
2. Non-decreasing: $2 \leq a_1 \leq a_2 \leq a_3 \leq a_4 \leq (n+1)/2$
3. Sub-additivities: $2a_1 \geq a_2$, $a_1 + a_2 \geq a_3$, $a_1 + a_3 \geq a_4$, $2a_2 \geq a_4$; $a_1 + 2a_4 \geq n+1$, $a_2 + a_3 + a_4 \geq n+1$, $3a_3 \geq n+1$,
4. Defining: $x_{i_1 - 1} + 1 \leq a_1 \leq x_{i_1}$, $x_{i_2 - 1} + 1 \leq a_2 \leq x_{i_2}$, $x_{i_3 - 1} + 1 \leq a_3 \leq x_{i_3}$, $x_{i_4 - 1} + 1 \leq a_4 \leq x_{i_4}$

where $x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4} \in X$ are the smallest indices $x \in X$ of $\xi_x \geq 1/8, 2/8, 3/8, 4/8$.

5 Worst case analysis

In the shooting experiment, we have observed a spike of the size of $1/k$-facets at $k = n$. In this section, we define the strength of a facet and analyze bounds on the strength of the $1/n$-facets introduced in Section 2.2.

5.1 Measures of facet size and strength

Although the facets of a polyhedron are all necessary elements of its description as a system of inequalities, there are ways in which some facets could be considered more important than others. One measurement for facet importance is the result of a shooting experiment, as described in Section 3. From the perspective of the origin (or an alternative point used as the origin of the shots), the shooting experiment measures the percent of the field of vision occupied by the individual facets; as this can be thought of as a measure of size from a given perspective for the remainder of the paper we will call facets that were frequently hit in the shooting experiment *large* and those that are not frequently hit *small*.

An alternative view of facet importance is known as the worst case analysis which, for an individual facet $\xi t \leq 1$ considers the problem of maximizing the objective ξt over the polyhedron after removal of the facet $\xi t \leq 1$. Since a facet is necessary for the description, the optimal objective value is always larger than one; however, if it is very close to one we may consider such facet to be *weak*, and if this value is larger we consider the facet to be *strong*.

Shim, Cao and Chopra [17] have computationally demonstrated that these notions of large and small, and strong and weak go together in the sense that these measures are highly correlated with each other, i.e. large facets were consistently found to be strong, and small facets were found to be weak. Since computing the strength or weakness of all individual facets is computationally intensive as n and the number of facets grow, the worst case analysis was only performed for values of $n \leq 26$. Their work is an important confirmation of the validity of the shooting experiment as an accurate measurement of facet importance for the master knapsack polytope; previously some had questioned the validity of shooting experiments because the result depends on the selection of a point from which to fire off the shots (the origin in our case). This confirmation is good news for the computational evaluation of facets because the shooting experiment is much more computationally tractable than the worst case analysis of all individual facets. In the following we define a slightly more general notion of strength and prove that it is equivalent to the notion of strength informally described above and used in [17].

5.2 Gradient lemma

Let $z^{IP}(v)$ denote the optimal value of the knapsack problem with objective $v \geq 0$ and let $z^{LP(\xi)}(v)$ denote the optimal value of the LP problem over the system of the knapsack equation, the non-negativity constraints and all knapsack facets except $\xi t \leq 1$.

Consider the primal linear programming problem over the system (20) describing $P(K(n))$ with

$\xi^1 t \leq 1$ deleted

$$\begin{array}{rl} \max & vt \\ st & lin(n) \cdot t = 1 \\ & \xi^1 t \leq 1 \\ & \vdots \\ & \xi^K t \leq 1 \\ & t \geq 0, \end{array} \tag{20}$$

where $lin(n) = (1/n, 2/n, ..., n/n)$ is the coefficient vector of the equation replacing the knapsack equation. Then, $z^{LP(\xi^1)}(v)$ is the optimal solution. Note that the dual problem can be written as

$$\min \quad x_0 + x_1 + ... + x_K, \tag{21}$$

$$st \quad x_0 \cdot lin(n) + x_1 \xi^1 + ... + x_K \xi^K - y = v, \tag{22}$$

$$x_1, ..., x_K \text{ and } y \text{ are all nonnegative.} \tag{23}$$

We fix $x_1 = 0$ when $\xi^1 t \leq 1$ is deleted.

The *LP-relaxation gap* of $\xi t \leq 1$ is defined to be

$$\max_{v \geq 0} \frac{z^{LP(\xi)}(v)}{z^{IP}(v)}.$$

Note that its appropriate to use max instead of sup above as $z^{LP(\xi)}(v)$ will always be bounded and have an optimal solution; this follows from the fact that any solution t satisfies $t \geq 0$ and $lin(n) \cdot t = 1$. A knapsack facet ξ is said to be *strong* if the gap is large and *weak* otherwise. In this section, we show that the gap is same as

$$z^{LP(\xi)}(\xi).$$

Lemma 5.1 *Let $\xi^k t \leq 1, k = 1, ..., K$, be the knapsack facets of $P(K(n))$ with $\xi^k_1 = 0$ and $\xi^k_n = 1$. Then, for each $k = 1, ..., K$, the LP-relaxation gap of ξ^k is*

$$\max_{v \geq 0} \frac{z^{LP(\xi^k)}(v)}{z^{IP}(v)} = \frac{z^{LP(\xi^k)}(\xi^k)}{z^{IP}(\xi^k)} = z^{LP(\xi^k)}(\xi^k). \tag{24}$$

Proof. Since $\xi^k_1 = 0$ for all $k = 1, ..., K$, the first component of the equation in (22) implies

$$x_0 = (v_1 + y_1)n \geq 0 \tag{25}$$

which can be added to (23) having all dual variables non-negative.

Let t^{IP} be an optimal solution to the primal problem (20) and let (x^{IP}, y^{IP}) be an optimal solution to the dual problem described in (21)-(23). Then, we have strong duality

$$vt^{IP} = \sum_{i=0}^{K} x_i^{IP}$$

and we may assume by scaling v that

$$z^{IP}(v) = vt^{IP} = \sum_{i=0}^{K} x_i^{IP} = 1. \tag{26}$$

Since all x_i^{IP} are non-negative in (23) and (25), equality (26) is followed by

$$x_i^{IP} \leq 1 \text{ for } i = 0, ..., K. \tag{27}$$

We only need to show the theorem for $k = 1$. If $\xi^1 t^{IP} < 1$, then $x_1^{IP} = 0$ by complementary slackness and eliminating $\xi^1 t \leq 1$ does not change the dual optimal value. We assume that $\xi^1 t^{IP} = 1$. Let t^{LP} be an optimal solution to the primal problem (20) with $\xi^1 t \leq 1$ eliminated. Then, we may assume that

$$\xi^1 t^{LP} \geq 1; \tag{28}$$

otherwise, t^{LP} is feasible for the original (20) and can be switched to $t^{LP} = t^{IP}$.

We complete the proof of the theorem by showing $\xi^1 t^{LP} \geq vt^{LP}$. From (27), (28) and (26), it follows that

$$\begin{aligned} \xi^1 t^{LP} &= (x_1^{IP} + (1 - x_1^{IP}))\xi^1 t^{LP} = x_1^{IP}\xi^1 t^{LP} + (1 - x_1^{IP})\xi^1 t^{LP} \\ &\geq x_1^{IP}\xi^1 t^{LP} + (1 - x_1^{IP})(1) = x_1^{IP}\xi^1 t^{LP} + \left(x_0^{IP} + \sum_{i=2}^{K} x_i^{IP} \right). \end{aligned} \tag{29}$$

Since $lin(n) \cdot t^{LP} = 1$ and $\xi^i t^{LP} \leq 1$ for $i = 2, ..., K$, (29) is followed by

$$\begin{aligned} \xi^1 t^{LP} &\geq x_1^{IP}\xi^1 t^{LP} + x_0^{IP} + \sum_{i=2}^{K} x_i^{IP} \\ &\geq x_1^{IP}\xi^1 t^{LP} + x_0^{IP}(lin(n) \cdot t^{LP}) + \sum_{i=2}^{K} x_i^{IP}(\xi^i t^{LP}) \\ &= x_0^{IP}(lin(n) \cdot t^{LP}) + x_1^{IP}\xi^1 t^{LP} + \sum_{i=2}^{K} x_i^{IP}(\xi^i t^{LP}) \\ &= x_0^{IP}(lin(n) \cdot t^{LP}) + \sum_{i=1}^{K} x_i^{IP}\xi^i t^{LP} \\ &\geq x_0^{IP}(lin(n) \cdot t^{LP}) + \sum_{i=1}^{K} x_i^{IP}\xi^i t^{LP} - y^{IP} t^{LP} \\ &= \left(x_0^{IP}(lin(n)) + x_1^{IP}\xi^1 + ... + x_K^{IP}\xi^K - y^{IP} \right) \cdot t^{LP} = vt^{LP}, \end{aligned}$$

completing the proof. □

5.3 The LP-relaxation gap of a $1/n$-facet

We now present bounds on the LP-relaxation gap of two classes of $1/n$-facets introduced in Section 2.2.

Theorem 5.2 *The LP-relaxation gap of $\xi^{n\text{-}(a_m)}$ given by $a_1 = ... = a_q = q$ and by $a_i = \lceil i \rceil$ for $q \le i \le n-q$ where $1 < q \le \frac{n}{4}$ satisfies*

$$z^{LP(\xi^{n\text{-}(a_m)})}(\xi^{n\text{-}(a_m)}) < 1 + \frac{q}{n} \le 1 + \frac{1}{4}.$$

Proof. Note that

$$\xi_i^{n\text{-}(a_m)} = \begin{cases} 0 & \text{for } i < q, \\ i/n & \text{for } q \le i \le n-q, and \\ 1 & \text{for } i > n-q. \end{cases}$$

Let $\hat{\xi}$ be the 1-facet with the shortest half landing. It holds that

$$\xi^{n\text{-}(a_m)} \le lin(n) + \frac{q-1}{n} \cdot \hat{\xi}$$

because for $i \le n-q$,

$$\xi_i^{n\text{-}(a_m)} - lin(n)_i = \xi_i^{n\text{-}(a_m)} - \frac{i}{n} \le \frac{i}{n} - \frac{i}{n} = 0 \le \frac{q-1}{n} \cdot \hat{\xi}_i,$$

and for $i \ge n-q+1$,

$$\xi_i^{n\text{-}(a_m)} - lin(n)_i \quad \le \quad 1 - \frac{i}{n} \le 1 - \frac{n-q+1}{n} = \frac{q-1}{n} = \frac{q-1}{n} \cdot \hat{\xi}_i.$$

From $q \le \frac{n}{4}$, it follows that

$$z^{LP(\xi^{n\text{-}(a_m)})}(\xi^{n\text{-}(a_m)}) \le 1 + \frac{q-1}{n} < 1 + \frac{q}{n} \le 1 + \frac{1}{4}.$$

□

Theorem 5.3 *The LP-relaxation gap of $\xi^{n\text{-}(a_m)}$ given by $a_i = i$ for $i < q$ even and $a_i = i+1$ for $i < q$ odd and by $a_i = i$ for $q \le i \le n-q$ where $q \le \frac{n}{2}$ is even and $q \le \frac{n-2}{3}$ if n is odd satisfies*

$$z^{LP(\xi^{n\text{-}(a_m)})}(\xi^{n\text{-}(a_m)}) \le \frac{n-q+2}{n-q+1} = 1 + \frac{1}{n-q+1} \le 1 + \frac{2}{n+2}.$$

Proof. With $\alpha = \frac{n-q+2}{n-q+1}$, it holds that

$$\alpha \cdot lin(n)_i = \alpha \cdot \frac{i}{n} \ge \frac{i}{n} + \frac{1}{n}$$

specially for $i = n-q+1$. Therefore,

$$\alpha \cdot lin(n) = \frac{n-q+2}{n-q+1} \cdot lin(n) \ge \xi^{n\text{-}(a_m)}.$$

From $q \le n/2$, it is followed by

$$\frac{n-q+2}{n-q+1} = 1 + \frac{1}{n-q+1} \le 1 + \frac{2}{n+2}.$$

□

The $1/n$-facets analyzed in Theorem 5.2 appear frequently in our shooting experiment. So, although the upper bound on their strength established in Theorem 5.2 does not guarantee that large or strong facets of this category exist for all n, it is consistent with our shooting experiment.

The upper bound of the LP-relaxation gap of $1/n$-facets in Theorem 5.3 proves that this second category of $1/n$-facets are weak asymptotically, because the right hand side of the bound approaches one as n approaches infinity. This result is consistent with our shooting experiment, where facets in this category were seldom observed and were not identified as large. Together, these results give us a deeper understanding of which $1/n$-facets are important.

6 Remarks

We have discussed separation and enumeration of $1/k$-facets for knapsack subproblems. We would hypothesize that these facets will be especially helpful on knapsack subproblems where the included indices L are somewhat evenly distributed among the values $\{1, 2, \ldots, n\}$. However, there are other cases in which there is reason to believe that the $1/k$-facets will be unhelpful. For example, if most or all of the indices L take relatively small values, then the $1/k$-facets will have some disadvantages for small values of k; this is because the superadditivity constraints imply that $\xi_i = 0$ for all $i \le \frac{n}{k+1}$ if ξ is a $1/k$-facet. Thus, the coefficients of the projected facet might be mostly, or entirely zero. Thankfully, in such extreme cases other techniques are likely to be effective, as we now explain.

Cyclic group relaxation. Gilmore and Gomory [7] observed a cyclic group repetition for the knapsack sub-problems with larger n. In particular, if the right hand size of the knapsack subproblem is considerably larger than all of the indices in the set L, then the cyclic group relaxation leads to especially useful inequalities.

Super-increasing knapsack. Super-increasing knapsack problems are a class of knapsack subproblems where $1/k$-facets with small k are not likely to work well. A super-increasing knapsack problem is a knapsack sub-problem (13) with respect to $L = \{i_1 < ... < i_l\}$ satisfying

$$\sum_{u<v} i_u \le i_v.$$

A subproblem (13) with $L = \{2^0, 2^1, ..., 2^{l-1}\}$ and $n = 2^l - 1$ is an example of super-increasing knapsack problem. In particular, super-increasing knapsack problems have the property that most of the indices in L are concentrated in the smaller values, likely making the $1/k$-facets ineffective for small values of k. It will be most interesting to identify strong facets of the convex hull of the integer solutions to a super-increasing knapsack problem. The recent work of Gupta [13] has recently studied super-increasing knapsack problems.

7 Conclusion

Our shooting experiment indicates that there is value to focusing on $1/k$-facets for small values of k, such as k dividing 6 or 8 when solving the master knapsack problem (and potentially single row relaxations of many integer program). For both the master problem and knapsack subproblems, $1/k$-facets are easier to enumerate and separate for small values of k, and for knapsack subproblems the number of projections of $1/k$-facets depends only on the size of the subproblem and not on the size n of the master problem. We have also studied the strength of two classes of $1/n$-facets, which were introduced by Aráoz et al. [2]; in agreement with our shooting experiment we found that one class is likely strong and the other is asymptotically weak.

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