A concise characterization of strong knapsack facets

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Abstract

For the integer knapsack problem, the $1/k$-facets with k dividing 6 or 8 were shown to be very strong and efficiently separated [4, 20]. We give a concise characterization of the $1/k$-facets for each k dividing 6, 8 in terms of a concise description of the coefficients. This allows these inequalities to be separated efficiently. We develop a reduced linear programming formulation to speed up identification of the facets hit during the shooting experiment, and confirm the strength of the $1/k$-facets with k dividing 6 or 8 in higher dimensional problems.

Keywords: integer programming; knapsack facets; shooting experiment; cyclic group problem; 2-slope facet

1. Introduction

A single row integer programming problem with non-negative integer variables, coefficients and right-hand side is known as an *integer knapsack problem.* The *master knapsack problem* $K(n)$ of order n is defined by

$$\begin{aligned} \text{maximize} \quad & vt && (1)\\ \text{subject to} \quad & \sum_{i=1}^{n} it_i = n && (2)\\ & t \geq 0 && (3)\\ & t_i \text{ are integers,} && (4) \end{aligned}$$

where t is a column vector of variables and v is a non-negative row vector. Observe that the *knapsack equation* (2) contains all possible coefficients from 1 to n. Thus, cases of the integer knapsack problem can be obtained from the master knapsack problem by leaving some variables out of the problem or fixing them to zero. Studies considering the structure of this problem typically ignore the objective function (1). The convex hull of the solutions to $K(n)$ is denoted by $P(K(n))$ and referred to as the *master knapsack polytope*. The dimension of $P(K(n))$ is shown in [19, 21] to be $n-1$, and the non-negativity constraints (3) are *facet-defining* (*i.e.*, inequalities necessary for the description of $P(K(n)))$ for $i \geq 2$. We call the non-negativity constraints *trivial* facets. The nontrivial facets are called *knapsack facets*. The convex hull of the solutions to an integer knapsack problem is a face of $P(K(n))$ where the excluded variables are fixed to be zero, namely

$$P(K(n)) \cap \{t : t_i = 0 \text{ for } t_i \text{ left out of (2)}\}.$$

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Thus, study of the knapsack facets is of interest not only for its own sake, but also to develop a deeper understanding of single row relaxations of many integer programming problems in general.

Since $P(K(n))$ is not full dimensional, each knapsack facet can be represented in infinitely many ways. Tilting the knapsack facets to be perpendicular to the non-negativity constraint $t_1 \geq 0$, a knapsack facet can be uniquely written as $\xi t \leq 1$ with $\xi_1 = 0$ and simply denoted by its coefficient vector ξ. All such coefficient vectors ξ are characterized by *super-additive relations* (8) as follows:

Theorem 1.1 (Aráoz [1], Hunsaker [15]) *The coefficient vectors ξ of the knapsack facets $\xi t \leq 1$ of $K(n)$ with $\xi_1 = 0$ are the extreme points of the system of linear constraints*

$$\xi_1 = 0, \tag{5}$$

$$\xi_n = 1, \tag{6}$$

$$\xi_i + \xi_{n-i} = 1 \qquad \text{for } 1 \leq i \leq n/2, \tag{7}$$

$$\xi_i + \xi_j \leq \xi_{i+j} \qquad \text{whenever } i + j < n. \tag{8}$$

The feasible solutions to the system give valid inequalities $\xi t \leq 1$ for $P(K(n))$.

Therefore, a knapsack facet ξ is a non-decreasing sequence because

$$\xi_i = 0 + \xi_i = \xi_1 + \xi_i \leq \xi_{i+1}.$$

An example of such a coefficient vector ξ is depicted in Figure 1.

The shape of a step function leads to a taxonomy of the knapsack facets. A knapsack facet $\xi t \leq 1$ is called a *$1/k$-facet* if k is the smallest possible integer such that

$$\xi_i \in \{0/k, 1/k, 2/k, ..., k/k\} \cup \{1/2\} \text{ for all } i = 1, ..., n. \tag{9}$$

Note that k is the least common multiple of the denominators of the irreducible fractions ξ_i other than $\xi_i = 1/2$.

The strength of $1/k$-facets for small values of k has been validated both theoretically and experimentally. In [20], Shim, Chopra and Cao showed that the removal of any $1/k$-facet for $k = 1, 3, 4$ weakens the LP-relaxation of a complete inequality description of $P(K(n))$ by a significant amount. They also showed that the removal of a $1/k$-facet for small values of k has a much more significant impact on the LP-relaxation than the removal of a $1/k$-facet for large values of k.

In [4], Chopra, Shim and Steffy used a shooting experiment to further confirm the importance of $1/k$-facets for small values of k. The shooting experiment works by selecting a random direction from the origin and then computing which facet is hit first when traveling in that direction. This process is repeated with many randomly selected directions, each of which is referred to as a *shot*; the intuition is that more frequently hit facets are more important. Shooting experiments for all values of $n \leq 150$ in [4] showed that that $1/k$-facets for $k = 1, 3, 4, 6, 8$ absorb over 75% shots fired. In other words, if objective function directions were randomly selected from the origin, in over 75% of cases, the facet hit would be a $1/k$-facet for $k = 1, 3, 4, 6, 8$.

For the master knapsack polytope, shooting in direction $v \geq 0$ is equivalent to maximizing $v\xi$ subject to constraints (5)-(8); the resulting LP is referred to as the *shooting LP*. Since Kuhn [17, 18] first proposed the shooting experiment to identify new facets of the TSP polytope, it has been used by many authors (Hunsaker [15], Gomory, Johnson and Evans [13], Dash and Günlük [6]) to study the size of each facet. For more details, we may refer to Hunsaker, Johnson and Tovey [16]. The natural assumption is that facets that are hit more frequently in a shooting experiment are likely to be more significant in any cutting plane approach.

Our goal in this paper is to contribute to both the theoretical as well as the experimental study of $1/k$-facets for small k. In order to elaborate on our contribution to the theoretical effort, it is useful to consider the 1/4-facet ξ for $P(K(29))$ shown in Figure 1. The figure shows the coefficients of the following inequality:

$$\sum_{i=6}^{11} \frac{1}{4} t_i + \sum_{i=12}^{17} \frac{2}{4} t_i + \sum_{i=18}^{23} \frac{3}{4} t_i + \sum_{i=24}^{29} t_i \leq 1$$

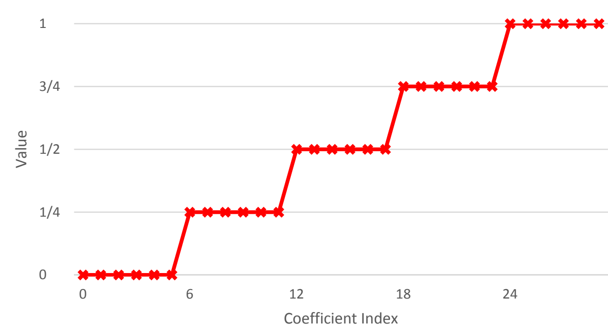

Figure 1: Coefficient vector of a knapsack facet in ξ-representation ($\xi t \leq 1$)

The x-axis in the figure provides the coefficient index from 1 to 29 and the y-axis provides the corresponding coefficient value. Observe that there are two ways to describe the inequality. One is by providing the coefficient for each variable ξ_i. The other is by providing the indices $a_1 = 6, a_2 = 12, a_3 = 18$, and $a_4 = 24$, where each a_m corresponds to the first index i such that $\xi_i \geq m/k$. $a_1 = 6$ corresponds to the facet index with coefficient $1/4$, $a_2 = 12$ corresponds to the first index with coefficient $2/4$ and so on. The inequality description using the sequence (a_m) is much more concise. Our first contribution in this paper is to provide a concise description (through the sequence (a_m)) for all $1/k$-facets, $k = 1, 3, 4, 6, 8$. The concise description allows these inequalities to be separated efficiently.

Our second contribution in this paper is to increase the size n of the knapsack problems for which the shooting experiment can be performed in practice. We do so by deriving a more concise description of the LP that must be solved for each shooting. Whereas Chopra, Shim, and Steffy [4] were able to perform the shooting experiment up to $n \leq 150$, our faster approach allows us to perform shooting experiments for n as large as 500. As a result we are able to confirm that $1/k$-facets for $k = 1, 3, 4, 6, 8$ (k dividing 6 or 8) absorb over 75% of the shots for n as large as 500.

Starting with Gomory [9], most of the work in the literature has focused on identifying facets of the form $\pi t \geq 1$ in the context of the group problem. In each case, effort has gone into identifying strong facets that can be described simply. Evans [7] and Aráoz, Evans, Gomory and Johnson [2] identified strong cyclic group and knapsack facets with a small number of slopes, which mainly come from 2-slope facets that were first constructed by Gomory and Johnson in [10, 11]. Cornuejols and

Molinaro [5] constructed 3-slope facets and Basu, Hildebrand, Koppe and Molinaro [3] constructed $(k+1)$-slope facets. Shim [19] and Shim and Johnson [21] focused on facets with fewer hills rather than slopes.

The paper is structured as follows. In Section 2 we derive a concise characterization of the $1/k$-facets for each k dividing 6 or 8. In Section 3 we develop a reduced linear programming formulation for the shooting problem in a sub-additive representation and discuss its relation with our super-additive representation. Section 4 evaluates the computational performance of this reduced formulation. We perform shooting experiments up to $n = 500$ to confirm that $1/k$-facets dividing 6 or 8 absorb over 75% of the shots.

2. Characterization of the strongest $1/k$-facets

Shim, Chopra and Cao [20] characterized and analyzed the 1-facets that Shim [19] and Shim and Johnson [21] first observed to be strong. Chopra, Shim and Steffy [4] defined $1/k$-inequalities that are observed to be strong for small values of k, independent of n. In particular, it was observed that, beyond the 1-facets, the $1/k$-facets are particularly strong when $k = 3, 4, 6$ or 8. In this section we obtain a concise description of these important facets.

A sequence $\xi = (\xi_i)_{i=1}^n$ is called *symmetric* if the complementarities (7) hold. We call $\xi t \leq 1$ a $1/k$-*inequality* if ξ is a non-decreasing symmetric sequence that satisfies (9). We see that a $1/d$-inequality is a $1/k$-inequality if d is a divisor of k. A $1/k$-inequality $\xi^{k\text{-}(a_m)}$ is uniquely determined by a non-decreasing sequence (a_m) where a_m represents the first index i with $\xi_i \geq m/k$ for $m \in \{0, 1, ..., k\} \cup \{k/2\}$. In Figure 1, the 1/4-facet ξ is a 1/4-inequality $\xi^{4\text{-}(a_m)}$ where $a_m = (a_1, a_2, a_3, a_4)$ is given by $(a_1 = 6, a_2 = 12, a_3 = 18, a_4 = 24)$.

Observe that the description of the inequality in terms of the sequence (a_m) is much more concise than the description in terms of all the coefficients of the variables ξ_i. The challenge, however, is to obtain the sequence (a_m) without having to explicitly obtain all the coefficients of the variables ξ_i. For $k = 3, 4, 6, 8$, the rest of this section provides concise set of inequalities that can be used to directly obtain the sequence (a_m).

The following lemma gives a polyhedral description of the necessary relationships between the elements of the sequence (a_m), defining the coefficients of a $1/k$-inequality $\xi^{k\text{-}(a_m)}$. As we will see, representing facets by these "break points" in the coefficient values of the facets, instead of considering the coefficient vectors ξ allows for a compact and convenient means for describing the facet structure.

Lemma 2.1 (Chopra, Shim and Steffy [4]) *A $1/k$-inequality $\xi^{k\text{-}(a_m)}$ satisfies (5)-(8) if and only if*

$$2 \leq a_{m_1} \leq a_{m_2} \leq (n+1)/2 \qquad \textit{for } m_1 \leq m_2 \leq k/2, \tag{10}$$

$$a_m + a_{k+1-\lceil m \rceil} = n + 1 \qquad \textit{for } m \leq k/2, \tag{11}$$

$$a_{m_1} + a_{m_2} \geq a_{\lceil m_1 + m_2 \rceil} \qquad \textit{for all } m_1 \leq m_2 \textit{ with } \lceil m_1 + m_2 \rceil \leq k. \tag{12}$$

The *index polytope* $P_k^n(a_m)$ is the bounded polyhedron of the indices (a_m) defined by (10)-(12). Note that the system has $O(k)$ variables and $O(k^2)$ constraints. (We could further reduce its size by substituting out $a_m, m > k/2$ using (11) and reduce the number of the variables to $\lceil k/2 \rceil$.) If k is fixed to be a constant, the number of variables and the number of constraints of the system are constant and independent of both n the size of the master problem and l the number of variables of a general knapsack problem.

This lemma is useful for a variety of purposes. First, given a $1/k$-inequality ξ we may use the above relations to verify whether or not it is a valid inequality for the knapsack polytope without needing to verify feasibility for the larger number of constraints given by (5)-(8). Second, if we are given a vector ξ representing a $1/k$-inequality for an integer knapsack problem with l non-zero variables, we can formulate an integer programming model to determine if values (a_m) satisfying (10)-(12) exist that correspond to ξ; in the absence of such a formulation, it is not clear how one might accomplish this check without attempting to extend ξ to a dense coefficient vector for the master knapsack problem that satisfies equations (5)-(8). Finally, and perhaps most useful, we will demonstrate how the relations (10)-(12) can be used to model the problem of finding a maximally violated valid $1/k$-inequality for a knapsack problem or subproblem.

Suppose we have an integer knapsack problem where $L \subseteq \{1, \dots, n\}$ is the index set of variables not fixed to zero and that $\hat{t}$ is a fractional solution to that knapsack problem (among other properties, $\hat{t} \geq 0$). Our goal is to find a valid $1/k$ inequality $\xi t \leq 1$ that maximizes ξt. The following formulation demonstrates how the separation problem can be formulated as a small integer programming problem. Here we let $K = \{1, \dots, k\} \cup \{k/2\}$ and introduce variables y_{ij} where (14) ensures that $a_j > i$ implies $y_{ij} = 0$. For simplicity of notation we assume in the following that k is even. If k is odd then (13) would be modified so that the term for $j = k/2$ and the following term in the series have coefficients of 1/2.

$$\max \quad \sum_{i \in L} \xi_i \hat{t}_i$$

$$\text{Subject to} \quad a \text{ satisfies (10)-(12)}$$

$$\xi_i = \sum_{j \in K} \frac{y_{ij}}{k} \quad \forall i \in L \tag{13}$$

$$y_{ij} \leq \frac{n - a_j}{n - i} \quad \forall i \in L, j \in K \tag{14}$$

$$a_j \text{ integer}, \; y_{ij} \text{ binary}$$

Thus if $l = |L|$, the separation problem for the integer knapsack problem becomes an integer programming problem with $O(kl)$ additional variables and constraints, beyond the $O(k)$ variables and $O(k^2)$ constraints from (10)-(12). Since all of this is independent of n, this will allow for tractable separation when k and l are small. In particular, if l, k are considered as fixed values, the separation can be done in constant time.

We now refine the inequalities given by Lemma 2.1 to precisely characterize the $1/k$-facets for each $k \in \{3, 4, 6, 8\}$ in terms of the sequence (a_m). Having a clear representation of which solutions of (10)-(12) correspond to facets will allow us to more easily enumerate these facets, or reduce the size of the separation integer programming problem listed above.

2.1. The 1/3-facets

In this section we characterize the coefficient structure of the $\frac{1}{3}$-facets. Note that using (11) it suffices to provide only the indices for coefficients not greater than $k/2$.

Theorem 2.2 *Let $\xi^{3\text{-}(a_1,a_{3/2})}$ be a 1/3-inequality with $\xi_1^{3\text{-}(a_1,a_{3/2})} = 0$, $\xi_n^{3\text{-}(a_1,a_{3/2})} = 1$, $a_1 < a_{3/2} \leq (n+1)/2$ and*

$$2a_1 + a_{3/2} \geq n + 1. \tag{15}$$

It is a knapsack facet, if and only if

$$3a_1 \leq n. \tag{16}$$

Proof. We first show that (15) implies the feasibility of $\xi^{3\text{-}(a_1,a_{3/2})}$ to the system (5)-(8). From Lemma 2.1, $\xi^{3\text{-}(a_1,a_{3/2})}$ satisfies (5)-(8), if and only if $(a_m) = (a_1, a_{3/2}, a_2, a_3) \in P_3^n(a_m)$; *i.e.*,

$$a_1 + a_3 = a_{3/2} + a_2 = n + 1, \tag{17}$$
$$a_1 + a_1 \geq a_2, \tag{18}$$
$$a_1 + a_{3/2} \geq a_3, \tag{19}$$
$$a_1 + a_2 \geq a_3. \tag{20}$$

From (19) and $a_{3/2} \leq a_2$, (20) follows trivially. Since (17) implies that (18) and (19) are equivalent to the single relation (15), $\xi^{3\text{-}(a_1,a_{3/2})}$ is valid satisfying all (5)-(8) if and only if (15) holds. (By definition, a 1/3-inequality is symmetric and satisfies the complementarities (17).)

We then show that (16) holds additionally if and only if $\xi^{3\text{-}(a_1,a_{3/2})}$ is an extreme point of the system (5)-(8). We only need to show that (16) holds additionally if and only if a feasible solution ξ to the system (5)-(8) satisfying the constraints binding at $\xi^{3\text{-}(a_1,a_{3/2})}$ at equality is uniquely determined to be $\xi = \xi^{3\text{-}(a_1,a_{3/2})}$. Before we prove the main result, we show that the value of all components of such a vector ξ are uniquely determined by the value ξ_{a_1}.

Suppose ξ is a feasible solution to the system (5)-(8) satisfying all the constraints binding at $\xi^{3\text{-}(a_1,a_{3/2})}$ at equality. Note that $\xi^{3\text{-}(a_1,a_{3/2})}$ satisfies the following super-additive relations at equality

$$\xi_1^{3\text{-}(a_1,a_{3/2})} + \xi_{i-1}^{3\text{-}(a_1,a_{3/2})} \leq \xi_i^{3\text{-}(a_1,a_{3/2})} \text{ for } i \notin \{a_1, a_{3/2}, a_2, a_3\}.$$

Therefore, ξ satisfies

$$\xi_{i-1} = 0 + \xi_{i-1} = \xi_1 + \xi_{i-1} = \xi_i \text{ for } i \notin \{a_1, a_{3/2}, a_2, a_3\}. \tag{21}$$

Therefore, ξ is determined by $\xi_{a_1}, \xi_{a_{3/2}}, \xi_{a_2}$ and ξ_{a_3}. Then, (5)-(7) and (21) imply that

$$\xi_{a_3} = 1 - \xi_{n-a_3} = 1 - \xi_{n-a_3-1} = ... = 1 - \xi_1 = 1 - 0 = 1, \tag{22}$$
$$\xi_{a_2} = 1 - \xi_{n-a_2} = 1 - \xi_{n-a_2-1} = ... = 1 - \xi_{a_1}, \text{ and} \tag{23}$$
$$\xi_i = 1/2 \text{ for } a_{3/2} \leq i \leq n - a_{3/2}. \tag{24}$$

Thus, ξ can be determined completely once we know ξ_{a_1}.

Note that $2a_1 \geq a_2$ due to (18). By (21), (16) holds (equivalently, $2a_1 \leq n + 1 - a_1 - 1 = a_3 - 1 < a_3$), if and only if $\xi_{2a_1} = \xi_{2a_1-1} = ... = \xi_{a_2}$. Moreover, $2\xi_{a_1} = \xi_{2a_1}$ holds as $2\xi_{a_1}^{3\text{-}(a_1,a_{3/2})} = 1/3 + 1/3 = 2/3 = \xi_{2a_1}^{3\text{-}(a_1,a_{3/2})}$. The three equations $\xi_{2a_1} = \xi_{a_2}$, $2\xi_{a_1} = \xi_{2a_1}$ and $\xi_{a_2} = 1 - \xi_{a_1}$ from (23) above, imply that $\xi_{a_1} = 1/3$ and thus $\xi = \xi^{3\text{-}(a_1,a_{3/2})}$. Therefore, $\xi^{3\text{-}(a_1,a_{3/2})}$ is an extreme point of the the system (5)-(8) and is a knapsack facet. □

2.2. The 1/4-facets

The 1/4-facets are characterized as follows:

Theorem 2.3 *Let $\xi^{4\text{-}(a_1,a_2)}$ be a 1/4-inequality with $\xi_1^{4\text{-}(a_1,a_2)} = 0, \xi_n^{4\text{-}(a_1,a_2)} = 1$, $a_1 < a_2 \leq (n+1)/2$,*

$$a_2 \leq 2a_1, \text{ and} \tag{25}$$
$$a_1 + 2a_2 \geq n + 1. \tag{26}$$

It is a knapsack facet, if and only if

$$2a_1 + a_2 \leq n. \tag{27}$$

Proof. We first show that (25) and (26) imply the feasibility of $\xi^{4\text{-}(a_1,a_2)}$ to the system (5)-(8). From Lemma 2.1, $\xi^{4\text{-}(a_1,a_2)}$ satisfies (5)-(8), if and only if $(a_m) = (a_1, a_2, a_3, a_4) \in P_4^n(a_m)$; *i.e.*,

$$a_1 + a_4 = a_2 + a_3 = n + 1, \tag{28}$$
$$a_1 + a_1 \geq a_2, \tag{29}$$
$$a_1 + a_2 \geq a_3, \tag{30}$$
$$a_1 + a_3 \geq a_4, \tag{31}$$
$$a_2 + a_2 \geq a_4, \tag{32}$$

Similar to the proof of Theorem 2.2, we see that (29) and (31) are equivalent to (25), and (30) and (32) are equivalent to (26).

We then show that (27) holds additionally if and only if $\xi^{4\text{-}(a_1,a_2)}$ is an extreme point of the system (5)-(8). We only need to show that (27) holds additionally if and only if a feasible solution ξ to the system (5)-(8) satisfying the constraints binding at $\xi^{4\text{-}(a_1,a_2)}$ at equality is uniquely determined to be $\xi = \xi^{4\text{-}(a_1,a_2)}$.

Suppose that ξ is a solution to (5)-(8) that is binding at the same inequalities as $\xi^{4\text{-}(a_1,a_2)}$. Observe that all components of ξ are uniquely determined based on the value of ξ_{a_1}. This follows from the fact that ξ satisfies

$$\xi_{i-1} = 0 + \xi_{i-1} = \xi_1 + \xi_{i-1} = \xi_i \text{ for } i \notin \{a_1, a_2, a_3, a_4\}, \tag{33}$$
$$\xi_{a_2} = 1/2 \tag{34}$$
$$\xi_{a_4} = 1 - \xi_1 = 1 - 0 = 1, \text{ and} \tag{35}$$
$$\xi_{a_3} = 1 - \xi_{a_1}. \tag{36}$$

Consider all possible cases of the evaluation of super-additive relations (8) binding at $\xi^{4\text{-}(a_1,a_2)}$:

$$0 + 0 = \xi_i^{4\text{-}(a_1,a_2)} + \xi_j^{4\text{-}(a_1,a_2)} = \xi_{i+j}^{4\text{-}(a_1,a_2)} = 0, \tag{37}$$
$$0 + 1/4 = \xi_i^{4\text{-}(a_1,a_2)} + \xi_j^{4\text{-}(a_1,a_2)} = \xi_{i+j}^{4\text{-}(a_1,a_2)} = 1/4, \tag{38}$$
$$0 + 1/2 = \xi_i^{4\text{-}(a_1,a_2)} + \xi_j^{4\text{-}(a_1,a_2)} = \xi_{i+j}^{4\text{-}(a_1,a_2)} = 1/2, \tag{39}$$
$$0 + 3/4 = \xi_i^{4\text{-}(a_1,a_2)} + \xi_j^{4\text{-}(a_1,a_2)} = \xi_{i+j}^{4\text{-}(a_1,a_2)} = 3/4, \tag{40}$$
$$0 + 1 = \xi_i^{4\text{-}(a_1,a_2)} + \xi_j^{4\text{-}(a_1,a_2)} = \xi_{i+j}^{4\text{-}(a_1,a_2)} = 1, \tag{41}$$
$$1/4 + 1/4 = \xi_i^{4\text{-}(a_1,a_2)} + \xi_j^{4\text{-}(a_1,a_2)} = \xi_{i+j}^{4\text{-}(a_1,a_2)} = 1/2, \tag{42}$$
$$1/4 + 1/2 = \xi_i^{4\text{-}(a_1,a_2)} + \xi_j^{4\text{-}(a_1,a_2)} = \xi_{i+j}^{4\text{-}(a_1,a_2)} = 3/4, \tag{43}$$
$$1/4 + 3/4 = \xi_i^{4\text{-}(a_1,a_2)} + \xi_j^{4\text{-}(a_1,a_2)} = \xi_{i+j}^{4\text{-}(a_1,a_2)} = 1, \tag{44}$$
$$1/2 + 1/2 = \xi_i^{4\text{-}(a_1,a_2)} + \xi_j^{4\text{-}(a_1,a_2)} = \xi_{i+j}^{4\text{-}(a_1,a_2)} = 1. \tag{45}$$

The equations $\xi_i + \xi_j = \xi_{i+j}$ corresponding to (37)-(41) include zero terms when evaluated by $\xi = \xi^{4\text{-}(a_1,a_2)}$. They are all induced by (33). Those corresponding to (44)-(45) are induced by complementarities (34) and (36). We only need to consider the non-trivial cases (42) and (43). They occur if and only if

$$\begin{aligned}&2\xi_{a_1}^{4\text{-}(a_1,a_2)} = 1/2 = \xi_{2a_1}^{4\text{-}(a_1,a_2)} < 3/4,\\ &\xi_{a_1}^{4\text{-}(a_1,a_2)} + \xi_{a_2}^{4\text{-}(a_1,a_2)} = 1/4 + 1/2 = 3/4 = \xi_{a_1+a_2}^{4\text{-}(a_1,a_2)} < 1.\end{aligned}$$

That is, ξ_{a_1} can be determined by

$$\begin{aligned}&\xi_{a_1} + \xi_{a_1} = \xi_{a_2} = 1/2, \text{ or}\\ &\xi_{a_1} + 1/2 = \xi_{a_1} + \xi_{a_2} = \xi_{a_3} = 1 - \xi_{a_1},\end{aligned}$$

if and only if one of the following two sub-additive relations are satisfied

$$\begin{aligned}&2a_1 \le n - a_2 = n + 1 - a_2 - 1 = a_3 - 1 < a_3\\ &a_1 + a_2 \le n - a_1 = n + 1 - a_1 - 1 = a_4 - 1 < a_4.\end{aligned}$$

Both of them are equivalent to (27), completing the proof of the theorem. □

2.3. The 1/6-facets and the 1/8-facets

In general, we characterize the $1/k$-facets $\xi^{k\text{-}(a_m)}$ in two steps. First, a $1/k$-facet $\xi^{k\text{-}(a_m)}$ should be a $1/k$-inequality feasible to (5)-(8). That is, $(a_m) \in P_k^n(a_m)$ satisfying (10)-(12). We then show that ξ which satisfies at equality the relations binding at $\xi^{k\text{-}(a_m)}$ is determined by break points ξ_{a_m} and ξ_{a_m} that are uniquely determined by a system of independent linear equations from super-additive relations binding at $\xi^{k\text{-}(a_m)}$.

For $k = 6$, we explicitly write (10)-(12) from Lemma 2.1 as follows:

Observation 2.4 A 1/6-inequality $\xi^{6\text{-}(a_m)}$ satisfies (5)-(8) if and only if it satisfies

1. Complementarities: $a_1 + a_6 = n+1$, $a_2 + a_5 = n+1$, $a_3 + a_4 = n+1$
2. Non-decreasing: $2 \leq a_1 \leq a_2 \leq a_3 \leq (n+1)/2$
3. Sub-additivities: $2a_1 \geq a_2$, $a_1 + a_2 \geq a_3$; $a_1 + 2a_3 \geq n+1$, $2a_2 + a_3 \geq n+1$.

The index polytope $P_6^n(a_m)$ for $k = 6$ is the bounded polyhedron defined by Relations 1-3 of indices $(a_m : m = 1, ..., 6)$ in Observation 2.4.

A 1/6-facet $\xi^{6\text{-}(a_m)}$ must satisfy $a_1 < a_2 < a_3$; *i.e.*, it must include two values $\xi_i^{6\text{-}(a_m)} = 1/6$ and $\xi_j^{6\text{-}(a_m)} = 2/6$. By definition of 1/6-facet, $\xi^{6\text{-}(a_m)}$ must satisfy $a_1 < a_2$; *i.e.*, it must include value $\xi_i^{6\text{-}(a_m)} = 1/6$. Suppose that ξ is a solution to (5)-(8) that is binding at the same inequalities as $\xi^{6\text{-}(a_m)}$. In the same manner as the proofs of Theorems 2.2 and 2.3, ξ can be determined by ξ_{a_1} and ξ_{a_2}. If $\xi_{a_1} = \xi_{a_2}$, there would be no binding super-additive relation to determine $\xi_{a_1} = 1/6$. Therefore, $\xi^{6\text{-}(a_m)}$ must satisfy $a_2 < a_3$; *i.e.*, $\xi_i^{6\text{-}(a_m)} = 1/3$ is required for deciding $\xi_{a_1} = 1/6$. Thus, we have that $a_1 < a_2 < a_3$.

We characterize the 1/6-facets in a_1, a_2, a_3 as follows.

Theorem 2.5 *Let $(a_m : m = 1, ..., 6)$ be an integer solution in $P_6^n(a_m)$ and let $a_1 < a_2 < a_3$. The 1/6-inequality $\xi^{6\text{-}(a_1,a_2,a_3)}$ is a knapsack facet if and only if $(a_m : m = 1, ..., 6)$ satisfies an additional pair of relations from the following three*

1. $2a_1 \leq a_3 - 1$,
2. $a_1 + a_2 \leq n - a_3$, *and*
3. $3a_2 \leq n$ *(equivalently,* $2a_2 \leq a_5 - 1$*).*

Proof. Suppose that ξ is a solution to (5)-(8) that is binding at the same inequalities as $\xi^{6\text{-}(a_m)}$. Then, ξ can be determined by ξ_{a_1} and ξ_{a_2}. We show that ξ_{a_1} and ξ_{a_2} are uniquely determined by the binding super-additive relations, if and only if $(a_m : m = 1, ..., 6)$ satisfies an additional pair of relations from Relations 1-3.

We may enumerate all the cases of binding super-additive relations including terms $\xi_i^{6\text{-}(a_1,a_2,a_3)} = 1/6$ or $\xi_j^{6\text{-}(a_1,a_2,a_3)} = 1/3$, and see that ξ_{a_1} and ξ_{a_2} are determined by any pair of cases among the following three

1. $1/6 + 1/6 = \xi_i^{6\text{-}(a_1,a_2,a_3)} + \xi_j^{6\text{-}(a_1,a_2,a_3)} = \xi_{i+j}^{6\text{-}(a_1,a_2,a_3)} = 1/3$,
2. $1/6 + 1/3 = \xi_i^{6\text{-}(a_1,a_2,a_3)} + \xi_j^{6\text{-}(a_1,a_2,a_3)} = \xi_{i+j}^{6\text{-}(a_1,a_2,a_3)} = 1/2$, and
3. $1/3 + 1/3 + 1/3 = \xi_i^{6\text{-}(a_1,a_2,a_3)} + \xi_j^{6\text{-}(a_1,a_2,a_3)} + \xi_{n-i-j}^{6\text{-}(a_1,a_2,a_3)} = 1$.

The three cases correspond to

1. $2\xi_{a_1} = \xi_{a_2}$,
2. $\xi_{a_1} + \xi_{a_2} = 1/2$, and
3. $3\xi_{a_2} = 1$,

respectively. Any pair of the three equations are independent and determine ξ_{a_1} and ξ_{a_2}, completing the proof of the theorem. □

In a similar manner, we characterize the 1/8-facets. We explicitly write (10)-(12) from Lemma 2.1 as follows:

Observation 2.6 A 1/8-inequality $\xi^{8\text{-}(a_m)}$ satisfies (5)-(8) if and only if it satisfies

1. Complementarities: $a_1 + a_8 = n + 1$, $a_2 + a_7 = n + 1$, $a_3 + a_6 = n + 1$, $a_4 + a_5 = n + 1$
2. Non-decreasing: $2 \leq a_1 \leq a_2 \leq a_3 \leq a_4 \leq (n + 1)/2$
3. Sub-additivities: $2a_1 \geq a_2$, $a_1 + a_2 \geq a_3$, $a_1 + a_3 \geq a_4$, $2a_2 \geq a_4$; $a_1 + 2a_4 \geq n + 1$, $a_2 + a_3 + a_4 \geq n + 1$, $3a_3 \geq n + 1$.

The index polytope $P_8^n(a_m)$ for $k = 8$ is the bounded polyhedron defined by Relations 1-3 of indices $(a_m : m = 1, ..., 8)$ in Observation 2.6.

The characterization of the 1/8-facets are more complicated than 1/6-facets in that there may be 1/8-facets missing 1/8 or 3/8 from the coefficient values ξ_i. We see that at least one of $\xi_i = 1/8$ and $\xi_i = 3/8$ comes up in the definition of 1/8-facets (otherwise we would have a 1/4-facet). We also see that a 1/8-facet ξ includes $\xi_i = 1/4$, without which it would not be uniquely determined because of lack of equality sub-additive relations. A 1/8-facet is in one of three cases; $a_1 < a_2 < a_3 < a_4$, $a_1 < a_2 < a_3 = a_4$ or $a_1 = a_2 < a_3 < a_4$.

We characterize the 1/8-facets ξ where all $\xi_i = 1/8, 2/8, 3/8$ show up as follows.

Theorem 2.7 *Let $(a_m : m = 1, ..., 8)$ be an integer solution in $P_8^n(a_m)$ and let $a_1 < a_2 < a_3 < a_4$. The 1/8-inequality $\xi^{8\text{-}(a_1,a_2,a_3,a_4)}$ is a knapsack facet, if and only if $(a_m : m = 1, ..., 8)$ satisfies an additional triple of relations from the following five*

1. $2a_1 \leq a_3 - 1$,
2. $a_1 + a_2 \leq a_4 - 1$,
3. $2a_2 \leq n - a_4$,
4. $a_1 + a_3 \leq n - a_4$, *and*
5. $a_2 + 2a_3 \leq n$.

Proof. Any triple of the five equations in ξ_{a_1}, ξ_{a_2} and ξ_{a_3} corresponding to the five relations in a_1, a_2, a_3 and a_4 are independent and determine ξ_{a_1}, ξ_{a_2} and ξ_{a_3}, completing the proof of the theorem. For more details, the readers may refer to the appendix. □

We characterize the remaining cases of the 1/8-facets as follows.

Theorem 2.8 *Let $(a_m : m = 1, ..., 8)$ be an integer solution in $P_8^n(a_m)$ and let $a_1 < a_2 < a_3 = a_4$. The 1/8-inequality $\xi^{8\text{-}(a_1,a_2,a_3,a_4)}$ is a knapsack facet, if and only if $(a_m : m = 1, ..., 8)$ satisfies three more additional constraints*

1. $a_3 = a_4 \leq n/2$,
2. $2a_1 \leq a_3 - 1 = a_4 - 1$, *and*
3. $2a_2 \leq n - a_4$.

Theorem 2.9 *Let $(a_m : m = 1, ..., 8)$ be an integer solution in $P_8^n(a_m)$ and let $a_1 = a_2 < a_3 < a_4$. The 1/8-inequality $\xi^{8\text{-}(a_1,a_2,a_3,a_4)}$ is a knapsack facet, if and only if $(a_m : m = 1, ..., 8)$ satisfies three more additional constraints*

1. $a_4 \leq n/2$,
2. $2a_2 + a_4 \leq n$, *and*
3. $a_2 + 2a_3 \leq n$.

3. Minimal representation of the shooting LP

The shooting LP to maximize $v\xi$ subject to constraints (5)-(8) may be reduced to a more concise description by maximizing $\sum_{1<i<n/2}(v_i - v_{n-i})\xi_i$ over the system of the following inequalities,

$$\xi_i + \xi_j \leq \xi_{i+j} \text{ for } i \leq j < i+j < n/2, \tag{46}$$

$$\xi_i + \xi_j + \xi_{n-i-j} \leq 1 \text{ for } i \leq j \leq n-i-j < n/2, \tag{47}$$

$$\text{and } 2\xi_{n/4} \leq \xi_{n/2} = \frac{1}{2} \text{ if } n \equiv 0 \mod 4. \tag{48}$$

In this section, the reduced linear programming formulation is transformed to a sub-additive representation and shown to have a minimal set of constraints which is a relaxation of the convex hull of the cyclic group facets.

Observe that because the knapsack polytope is not full dimensional, facets can be represented either using the super-additive ξ-representation (characterizing facets $\xi t \leq 1$ as discussed in Sections 1 and 2) or using the sub-additive relations among π variables for characterizing the facets in the form $\pi t \geq 1$. In Fulkersons blocking framework of the cyclic group and the knapsack polyhedra, Gomory [9] and Aráoz [1] preferred the sub-additive characterization of these facets. We now develop the relationship between the cyclic group problem and the knapsack problem for the π-representation and use it to obtain a minimal representation of the shooting LP.

The *cyclic group problem* (C_n, b) with respect to a cyclic group C_n of order n and a non-zero element b, has a feasible region given by vectors t that satisfy

$$\sum_{g \in C_n \setminus \{0\}} g t_g = b,$$

where t_g, $g \in C_n \setminus \{0\}$, are non-negative integer variables. The convex hull of the integer solutions t of the problem was shown by Gomory [9] to be a polyhedron and is referred to as the *cyclic group polyhedron*, denoted by $P(C_n, b)$. The non-negativity constraints $t_g \geq 0$, $g \neq 0$, are facets of $P(C_n, b)$ and are called *trivial* facets. The non-trivial facets are denoted by

$$\pi t \geq \pi_b$$

and called *cyclic group facets*. Since $P(C_n, b)$ is known in [9] to be full dimensional, the cyclic group facets are uniquely represented on fixing $\pi_b = 1$.

Let $lin(n) = (1/n, 2/n, ..., n/n = 1)$ be the coefficient vector of the knapsack equation (2) normalized by the right-hand side n. We call it *lineality*. (We also refer to the linear space L generated by $lin(n)$ as *lineality*.) The knapsack equation $lin(n)t = 1$ is a cyclic group facet of the master group polyhedron denoted by $P(C_{n+1}, n)$, which Gomory [9] called the *mixed integer cut*, and the knapsack facets $\pi t \geq 1$ of $P(K(n))$ are the cyclic group facets adjacent to the mixed integer cut. That is, $P(K(n))$ is a cyclic group facet $lin(n)$ for $P(C_{n+1}, n)$ and the cyclic group facets adjacent to $lin(n)$ for $P(C_{n+1}, n)$ are the knapsack facets for $P(K(n))$. The nontrivial facets of $P(C_{n+1}, n)$ are the extreme points π of the bounded polyhedron

$$\Pi(C_{n+1}, n) = \Big\{\pi \in \mathbb{R}^{C_{n+1}-\{0\}} : \quad \pi_n = 1, \tag{49}$$

$$\pi \geq 0, \tag{50}$$

$$\pi_i + \pi_j = \pi_n \text{ if } i+j \equiv n \mod n+1, \tag{51}$$

$$\pi_i + \pi_j \geq \pi_k \text{ if } i+j \equiv k \mod n\Big\}, \tag{52}$$

where none of i, j, k is n. The nonnegativity constraints (50) are known in [21] to be redundant.

Aráoz [1] first characterized the knapsack facets in sub-additive relations. The *polar cone* $S(K(n)) \subseteq \mathbb{R}^n$ of the knapsack facets for $P(K(n))$ is given by the *complementarity constraints* (53) and the *sub-additive relations* (54),

$$\pi_i + \pi_{n-i} = \pi_n \text{ for } i = 1, ..., \left\lfloor \frac{n}{2} \right\rfloor, \tag{53}$$

$$\pi_i + \pi_j \geq \pi_{i+j} \text{ whenever } i + j < n, \tag{54}$$

and a defining set of rays π of $S(K(n))$ is shown in [1] to represent all the knapsack facets

$$\sum_{i=1}^{n} \pi_i t_i \geq \pi_n.$$

The knapsack facets $\xi = -\pi$ can be obtained from the extreme points π of the polar cone $\Pi^{\xi}(K(n))$ fixing $\pi_1 = 0$ and $\pi_n = -1$; *i.e.*,

$$-\xi = \pi \in \Pi^{\xi}(K(n)) = S(K(n)) \cap \{\pi : \pi_1 = 0\} \cap \{\pi : \pi_n = -1\}.$$

Thus, $-\xi$ is in the extreme ray of the translated cone $S(K(n)) \cap \{\pi : \pi_n = -1\}$ going through $-lin(n)$. Although $\Pi^{\xi}(K(n))$ has polynomially many constraints, it has exponentially many extreme points and the extreme points π correspond to the knapsack facets $\pi t \geq -1$. (Group facets and knapsack facets are known in Gomory-Johnsons papers (for example, Example 4.6 of [11]) to be exponentially many.)

Shooting in the direction $v \geq 0$ is equivalent to maximizing $v\xi$ subject to (5)-(8), or solving the shooting LP

$$\min \left\{ v\pi : \pi \in \Pi^{\xi}(K(n)) = S(K(n)) \cap \{\pi : \pi_1 = 0\} \cap \{\pi : \pi_n = -1\} \right\}. \tag{55}$$

The optimal solution $\xi = -\pi$ to the problem is the knapsack facet $\xi t \leq 1$ hit by shooting in the direction $v \geq 0$; *i.e.*, if we shoot an arrow at the knapsack facets in direction v from the origin, the facet $\xi t \leq 1$ is the first facet hit by the arrow. The system of relations (53)-(54) defining the polar cone include many redundant relations. By deleting the redundant relations, we can identify a minimal set of relations to describe $S(K(n))$ and develop a fast shooting linear programming formulation.[4]

To identify a minimal description of $S(K(n))$, we use the sub-additive representation $\pi t \geq 1$ of the knapsack facets fixing $\pi_n = 1$ and define the *natural inner point*

$$\dot{\pi} = (\dot{\pi}_i = 1/2 \text{ for } i \neq n; \dot{\pi}_n = 1)$$

which satisfies all the sub-additive relations (54) as strict inequalities. In the following we provide a minimal description of $\Pi(K(n)) = S(K(n)) \cap \{\pi : \pi_1 = 1/2\} \cap \{\pi : \pi_n = 1\}$ moving the natural inner point $\dot{\pi}$ out of each necessary relation. Note that $\Pi^{\xi}(K(n))$ and $\Pi(K(n))$ are translated on $S(K(n)) \cap \{\pi : \pi_n = 0\}$ and scaled to satisfy

$$\left(\Pi^{\xi}(K(n)) + lin(n)\right)\left(\frac{1}{2} - \frac{1}{n}\right) = \frac{1}{n}\left(\Pi(K(n)) - lin(n)\right). \tag{56}$$

[4] In this paper, a *minimal* representation (or description) is not minimal as in *minimal* inequalities defined by Gomory and Johnson [10] but in a *minimal* set of constraints required for $S(K(n))$.

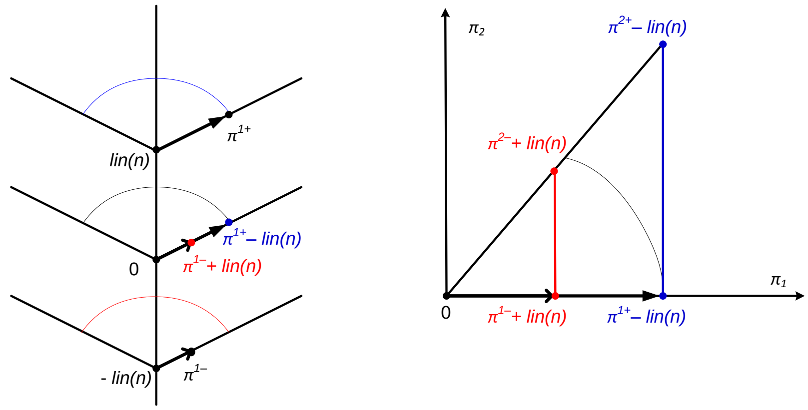

Figure 2: The restriction of $S(K(6))$ on (π_1, π_2, π_6)-space

In Figure 2, $\Pi^\xi(K(6))$ and $\Pi(K(6))$ are translated and lie on the cone $S(K(6)) \cap \{\pi : \pi_6 = 0\}$ pointed at the origin. The 3-dimensional cone on the left depicts $S(K(6))$ containing three parallel 2-dimensional layers defined by $\{\pi : \pi_n = 1\}$, $\{\pi : \pi_n = 0\}$ and $\{\pi : \pi_n = -1\}$. The two extreme rays π of each layer represent the knapsack facets for $P(K(6))$ in $\pi t \geq \pi_n$. On the right, the figure depicts two parallel line segments in the cone which are

$$\begin{aligned}
&\Pi(K(6)) - lin(6) = [\pi^{1+} - lin(6), \pi^{2+} - lin(6)] \\
&= \left[\left(\frac{1}{2} - \frac{1}{6}, \frac{1}{3} - \frac{2}{6}, \frac{1}{2} - \frac{3}{6}, \frac{2}{3} - \frac{4}{6}, \frac{1}{2} - \frac{5}{6}, 1 - \frac{6}{6}\right), \left(\frac{1}{2} - \frac{1}{6}, 1 - \frac{2}{6}, \frac{1}{2} - \frac{3}{6}, 0 - \frac{4}{6}, \frac{1}{2} - \frac{5}{6}, 1 - \frac{6}{6}\right)\right] \\
&= \left[\left(\frac{1}{3}, 0, 0, 0, -\frac{1}{3}, 0\right), \left(\frac{1}{3}, \frac{2}{3}, 0, -\frac{2}{3}, -\frac{1}{3}, 0\right)\right],
\end{aligned}$$

$$\begin{aligned}
&\Pi^\xi(K(6)) + lin(6) = [\pi^{1-} + lin(6), \pi^{2-} + lin(6)] \\
&= \left[\left(-0 + \frac{1}{6}, -\frac{1}{3} + \frac{2}{6}, -\frac{1}{2} + \frac{3}{6}, -\frac{2}{3} + \frac{4}{6}, -1 + \frac{5}{6}, -1 + \frac{6}{6}\right),\right. \\
&\qquad \left.\left(-0 + \frac{1}{6}, -0 + \frac{1}{3}, -\frac{1}{2} + \frac{3}{6}, -1 + \frac{4}{6}, -1 + \frac{5}{6}, -1 + \frac{6}{6}\right)\right] \\
&= \left[\left(\frac{1}{6}, 0, 0, 0, -\frac{1}{6}, 0\right), \left(\frac{1}{6}, \frac{1}{3}, 0, -\frac{1}{3}, -\frac{1}{6}, 0\right)\right].
\end{aligned}$$

For more details of parallel translation, the readers may refer to Shim [19].

The next theorem provides a minimal representation of $\Pi(K(n))$.

Theorem 3.1 *A minimal representation of $\Pi(K(n))$ is the system described by $\pi_1 = 1/2$, $\pi_n = 1$ and the complementary constraints (53) along with the inequalities (54) replaced by*

$$\pi_i + \pi_j \geq \pi_{i+j}, \ \textit{for } i \leq j < i + j < n/2, \tag{57}$$

$$\pi_i + \pi_j + \pi_{n-i-j} \geq \pi_n, \ \textit{for } i \leq j \leq n - i - j < n/2, \tag{58}$$

$$\textit{and } 2\pi_{n/4} \geq \pi_{n/2} = \frac{1}{2}, \ \textit{if } n \equiv 0 \mod 4. \tag{59}$$

In the following, we develop results that imply Theorem 3.1, which we later restate as Corollary 3.5. From (56), we see that the shooting LP (55) is equivalent to

$$\min\Big\{v\pi : \pi \in \Pi(K(n)) = S(K(n)) \cap \{\pi : \pi_1 = 1/2\} \cap \{\pi : \pi_n = 1\}\Big\}. \tag{60}$$

To identify a minimal description of $S(K(n))$, assuming $n \geq 3$, we define the index sets for the left-hand sides of complementarities (53) with indices $< n/2$ by O and with indices $> n/2$ by X; i.e.,

$$O = \{i : i < n/2\},$$
$$X = \{n - i : i < n/2\}.$$

We project out the X-variables by substituting $\pi_{n-i} = \pi_n - \pi_i$ from (53) into the constraints in (54) for $i < n/2$.

Theorem 3.2 *The two types of inequalities (57), (58) and the additional inequality (59) in case of n divisible by 4 completely describe $S(K(n))$ with the complementarities (53).*

Proof.
If n is even, we distinguish the *half* $h = n/2$ and also project out π_h by substituting the complementarity constraint

$$2\pi_h = \pi_n. \tag{61}$$

Then, $S(K(n))$ is equivalent to the projected image

$$S(K(n))|_{O\cup\{n\}} \subseteq \mathbb{R}^{O\cup\{n\}}$$

onto $\mathbb{R}^{O\cup\{n\}}$.

If n is odd, each knapsack subadditivity constraint (54) becomes one of (57) or (58) If n is even, the knapsack subadditivity constraints (54) without any term of π_h are of the two types (57) and (58). Any subadditivity constraints containing π_h have the form

$$\pi_i + \pi_{h-i} \geq \pi_h = \pi_n/2,$$

which is redundant if $i > h - i$ because it can be written as the average of $2\pi_{h-i} \geq \pi_{n-2i}$ and $2\pi_i + \pi_{n-2i} \geq \pi_n$. We therefore have at most one *additional* inequality (59).

□

Theorem 3.2 reduces the shooting LP as follows:

Theorem 3.3 *Let $v \geq 0$ and let $w_i = v_i - v_{n-i}$ for $1 < i < n/2$. The restriction $\pi|_{O\setminus\{1\}}$ of the optimal solution π to (60) is the optimal solution to*

$$\min\left\{\sum_{i\in O\setminus\{1\}} w_i\pi_i : (\pi_i : i \in O \setminus \{1\}) \in \Pi(K(n))|_{O\setminus\{1\}}\right\}, \tag{62}$$

where $\Pi(K(n))|_{O\setminus\{1\}}$ is described by fixing $\pi_1 = 1/2$ and $\pi_n = 1$ by (57) and (58) and by (59) if n is a multiple of 4.

The number of constraints (57)-(59) is much smaller than the number of constraints (54) and the number of variables of (62) is only a half of that of (60). Thus solving (62) described by (57)-(59) will be faster than solving (60) described by (54). We verify this experimentally in Section 4.

Since $\dot{\pi}$ satisfies (54) as strict inequalities, the projected cone $S(K(n))|_{O\cup\{n\}}$ is full dimensional with the restriction $\dot{\pi}|_{O\cup\{n\}}$ of $\dot{\pi}$ to $O \cup \{n\}$ as an interior point,

$$\dot{\pi}|_{O\cup\{n\}} = (\dot{\pi}_i = 1/2 \text{ for } i < n/2; \dot{\pi}_n = 1).$$

Therefore the dimension of $S(K(n))$ is $\lceil n/2 \rceil$. By considering modifications of $\dot{\pi}|_{O\cup\{n\}}$ we show that the two types of inequalities (57), (58) and the additional inequality (59) are facet-defining for $S(K(n))|_{O\cup\{n\}}$, and therefore form a minimal description of $S(K(n))$ together with the complementarity constraints (53). We say that $\hat{\pi}$ is a *certificate* for a constraint of $S(K(n))|_{O\cup\{n\}}$ to be facet-defining if $\hat{\pi}$ violates that constraint, but satisfies all other constraints. The intersection point of the line segment between $\hat{\pi}$ and $\dot{\pi}$ with the hyperplane of the constraint satisfies the constraint as equality and the other constraints as strict inequality.

Theorem 3.4 *Each inequality in (57)-(59) is facet-defining for $S(K(n))$.*

Proof. For $\pi_{i_0} + \pi_{j_0} \geq \pi_{i_0+j_0}$ with $i_0, j_0, i_0 + j_0 \in O$, we change at most three components of $\dot{\pi}$ to get an infeasible solution

$$\hat{\pi} = (\hat{\pi}_{i_0} = \hat{\pi}_{j_0} = 1/3, \hat{\pi}_{i_0+j_0} = 3/4),$$

where the other components remain the same as the corresponding components of $\dot{\pi}$. The infeasible solution $\hat{\pi}$ is shown to be feasible for all the other inequalities in (57):

$$\begin{aligned} \hat{\pi}_i + \hat{\pi}_j &= \frac{1}{3} + \frac{1}{3} \geq \frac{1}{2} \geq \hat{\pi}_{i+j} \qquad && \text{if } i_0 \neq j_0, i = j = i_0, \\ \hat{\pi}_i + \hat{\pi}_j &\geq \frac{1}{3} + \frac{1}{2} \geq \frac{3}{4} \geq \hat{\pi}_{i+j} && \text{otherwise.} \end{aligned}$$

It is also feasible for all the inequalities in (58):

$$\hat{\pi}_i + \hat{\pi}_j + \hat{\pi}_{n-i-j} \geq \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1 = \hat{\pi}_n.$$

The additional inequality (59) is trivially feasible.

For $\pi_{i_0} + \pi_{j_0} + \pi_{k_0} \geq \pi_n$ with $i_0 \leq j_0 \leq k_0$ and $i_0 + j_0 + k_0 = n$, we set $\hat{\pi}_{j_0} = 1/4$ in case $i_0 = j_0 = k_0 = n/3$. Otherwise, we set

$$\hat{\pi} = \left(\hat{\pi}_{j_0} = \frac{1}{3};\ \hat{\pi}_{i_0} = \frac{1}{4} \text{ if } i_0 \neq j_0;\ \hat{\pi}_{k_0} = \frac{1}{4} \text{ if } k_0 \neq j_0\right)$$

where the other components are same as the corresponding components of $\dot{\pi}$. It is infeasible for the inequality and feasible for all the inequalities in (57). Since all the other inequalities in (58) contain at least one term equal to 1/2, the point $\hat{\pi}$ is feasible for all of them. The inequality (59) is trivially feasible here again.

In case n is a multiple of 4, the additional inequality (59) has $\hat{\pi}_{n/4} = 0$ in which the other components are the same as the corresponding components of $\dot{\pi}$. It is trivially infeasible for (59) and feasible for all the inequalities in (57). Since $\pi_{n/4}$ comes up at most once in the left-hand side of each inequality in (58), $\hat{\pi}$ is feasible for all the inequalities, completing the proof of Theorem 3.4. □

Thus, we arrive at the following minimal description of $S(K(n))$, which directly implies our desired result, Theorem 3.1.

Corollary 3.5 *The two types of inequalities (57), (58) and the additional inequality (59) in case of n divisible by 4 form a minimal description of $S(K(n))$ with the complementarities (53).*

We now compare the number of constraints from (57)-(59) to that of constraints (54) and see that the ratio is asymptotically 1/3. We can calculate the asymptotic ratio as follows: The number of constraints (54) is

$$\sum_{2\le k<n} |\{i : 1 \le i \le j < k = i + j < n\}|$$
$$= \sum_{2\le k<n} \left|\left\{i : 1 \le i \le \left\lfloor \frac{k}{2} \right\rfloor\right\}\right| = \sum_{2\le k<n} \left\lfloor \frac{k}{2} \right\rfloor \sim \frac{n^2}{4}.$$

The number of constraints (57) is

$$\sum_{2\le k<n/2} |\{i : 1 \le i \le j < k = i + j < n/2\}|$$
$$= \sum_{2\le k<n/2} \left|\left\{i : 1 \le i \le \left\lfloor \frac{k}{2} \right\rfloor\right\}\right| = \sum_{2\le k<n/2} \left\lfloor \frac{k}{2} \right\rfloor \sim \frac{n^2}{16},$$

and the number of constraints (58) is

$$\sum_{n/3\le k<n/2} |\{j : 1 \le i \le j \le k \text{ and } i + j + k = n\}|$$
$$= \sum_{n/3\le k<n/2} \left|\left\{j : \left\lceil \frac{n-k}{2} \right\rceil \le j \le k\right\}\right| = \sum_{n/3\le k<n/2} \left(k - \left\lceil \frac{n-k}{2} \right\rceil + 1\right) \sim \frac{n^2}{48}.$$

Thus, the asymptotic ratio is $1/3 = (1/4) + (1/16 + 1/48)$.[5]

The number of variables of the reduced shooting LP (62) is half that of the original shooting LP (60) and the number of constraints is one third of the original. As we confirm computationally in the following section, Theorem 3.1 allows us to perform the shooting experiment more efficiently and thus enables computations for larger values of n.

4. Computations with the reduced shooting LP

One motivation for finding a minimal description for $S(K(n))$ is to improve the speed of shooting experiments. We now describe computational tests used to evaluate the magnitude of this

[5] The asymptotic ratio is provided by Kangbok Lee at POSTECH, South Korea.

	Time for 1000 shots (sec.)	
Dimension	Reduced formulation	Original formulation
$n = 100$	3.5	8.2
$n = 150$	9.2	23.1
$n = 200$	19.5	50.2
$n = 250$	36.6	101.3
$n = 300$	61.8	180.4

Table 1: Comparison of computation time required for 1000 shots.

improvement in practice. We refer to the formulation for $S(K(n))$ given by (53) and (54) (as in [1]) as the *original formulation*, and the minimal description from Corollary 3.5 as the *reduced formulation*.

For the shooting experiment, we optimize a linear objective function over the feasible region given by $S(K(n))$. The random objective function $v \geq 0$ is chosen from the non-negative orthant uniformly at random with respect to the n-dimensional sphere. In the reduced formulation v is mapped to the modified objective function w as described in Theorem 3.3. We compare the original and reduced formulations for various sizes of n, and measure the time required to evaluate 1000 shots.

All experiments were performed on a computer with a 3.4Ghz Intel i7 CPU running Ubuntu 12.04 LTS Linux. The Python interface of the Gurobi (version 5.0.2) LP solver was used for all experiments. In studies such as [4, 20] it is often desirable to perform a greater number of shots, such as one million, but 1000 shots are sufficient to compare the running time of the two formulations, which is the goal of our test. We note that when multiple shots are performed on $S(K(n))$ for the same value of n, the LPs solved for different shots share the same constraints and differ only in their objective function. This applies to both formulations. Therefore we update the model by changing the objective function and reoptimize. This allows the solver to take advantage of the existing construction of the constraints and use warm-start information from the previous shot to accelerate the solution process.

Solution times are presented in Table 1 and in Figure 3. The values of n range from 100 up to 300, in increments of 10. (Table 1 lists a subset of n's in increments of 50.) Times are given in seconds and represent the time to solve 1000 shooting LPs. We observe that the reduced formulation gives a significant performance increase over the original formulation, with a speedup factor of approximately three on the larger models.

We use the reduced formulation to perform the shooting experiment with a million shots each for n as large as 500 (see Figure 4). Our experiments confirm that $1/k$-facets for k dividing 6 or 8 absorb over 75% of a million shots (an observation first made in [4] for $n \leq 150$). The strength of these facets becomes particularly useful because they can be separated efficiently as shown in Section 3. Our results suggest that $1/k$-facets for k dividing 6 or 8 may be effective in practice when solving knapsack problems.

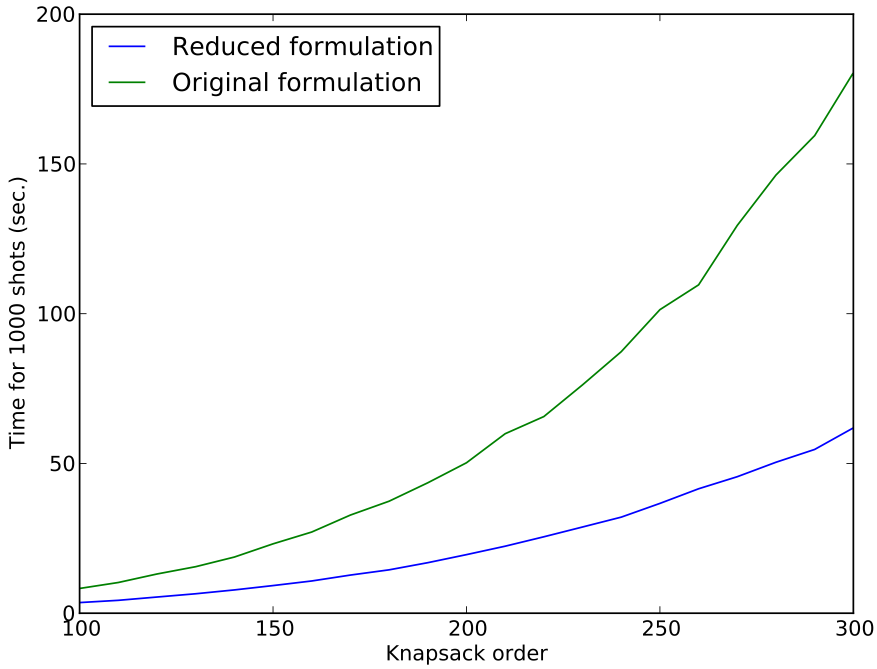

Figure 3: Comparison of computation time required for 1000 shots.

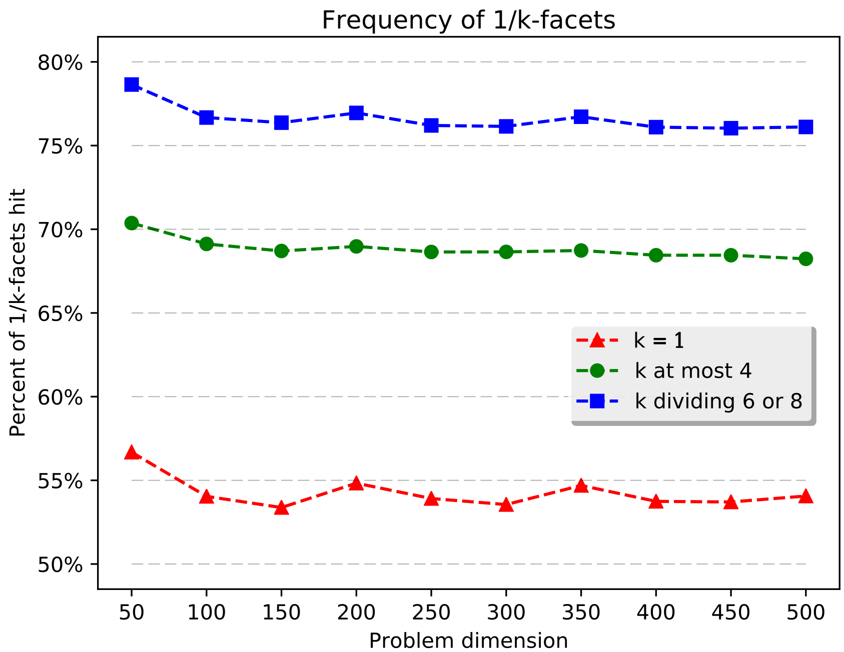

Figure 4: Results from fast shooting experiments for large dimension

5. Relating super-additive and sub-additive characterizations of knapsack facets

In this section, we show the relationship between the sub-additive representation ($\pi t \geq 1$) of a knapsack facet and the super-additive representation $\xi t \leq \xi_n$. The complementarity constraints (51) are exactly (53) and the sub-additive relations (52) include the sub-additive relations (54) of $S(K(n))$ which we will call *knapsack sub-additive relations*. The sub-additive relations (52) which are not in (54) can be written as

$$\pi_i + \pi_j \geq \pi_{i+j-n-1} \text{ whenever } i + j > n + 1, \tag{63}$$

which we will call *non-knapsack sub-additive relations*. Thus, $\Pi(C_{n+1}, n)$ is contained in the translated cone

$$S(K(n)) \cap \{\pi : \pi_n = 1\} = S(K(n)) \cap \{\pi : \pi_n = 0\} + lin(n).$$

In fact, $\Pi(C_{n+1}, n)$ is known in [19] to be full dimensional in $S(K(n)) \cap \{\pi : \pi_n = 1\}$.

In this paper, we characterized the knapsack facets by super-additive relations of $\xi = -\pi$ representing a knapsack facet in $\xi t \leq \xi_n$ with fixing $\xi_1 = 0$ and $\xi_n = 1$. Hunsaker [15] first described the knapsack facets by the non-trivial facets of the packing knapsack problem defined by (2)-(4) with equation (2) replaced by the packing inequality

$$\sum_{i=2}^{n} it_i \leq n.$$

It is equivalent to fixing the coefficient $\xi_1 = 0$ of a knapsack facet $\xi t \leq \xi_n$ as t_1 is a slack variable.

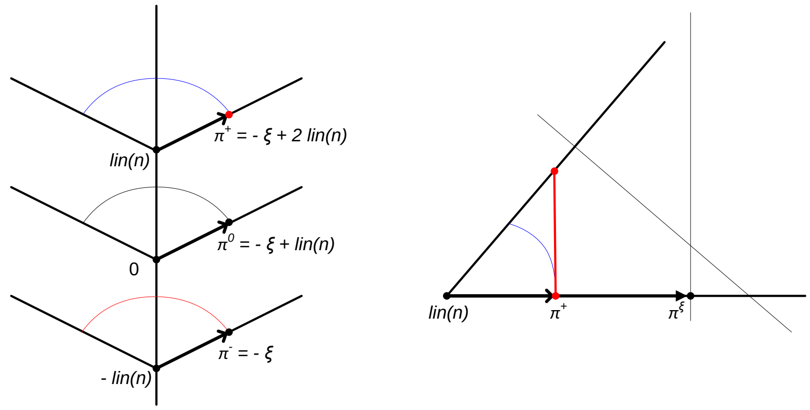

Figure 5: $S(K(6))$ and $\Pi(C_7, 6) \subset S(K(6)) \cap \{\pi : \pi_6 = 1\}$

In the remainder of this section, we geometrically show in the dual space, so called π-space, how to transform knapsack facets and cyclic group facets interchangeably in order to invoke strong knapsack facets from previously known strong cyclic group facets, as Aráoz, Evans, Gomory and Johnson [2] identified knapsack facets from cyclic group facets. It is shown by Shim [19] and Chopra, Shim and Steffy [4] that an extreme point π of $\Pi(C_{n+1}, n)$ is adjacent to extreme point $lin(n)$ if and only if cyclic group facet $\pi t \geq 1$ is adjacent to $lin(n)t \geq 1$. It allows exploring the relation between the knapsack facets for $P(K(n))$ and the cyclic group facets for $P(C_{n+1}, n)$ as points π in the space containing the polar cone $S(K(n))$, which Gomory, Johnson and Evans [13] referred to as π-space. For more details of the adjacency of group facets, we refer to Tyber and Johnson [23].

Since the cyclic group facets for $P(C_{n+1}, n)$ which are adjacent to $lin(n)$ are known in [2] to be the knapsack facets for $P(K(n))$, the extreme points π adjacent to $lin(n)$ in $\Pi(C_{n+1}, n)$ represent the knapsack facets $\pi t \geq 1$. Therefore, we can transform a knapsack facet ξ to the corresponding cyclic group facet π^ξ in π-space by shooting[6] from $lin(n)$ at the translate $\pi^+ = -\xi + 2lin(n)$ of $\pi^- = -\xi$ (or along the edge from $lin(n)$ through π^+) and seeing the hitting point π^ξ where a non-knapsack sub-additive relation is first hit. The first hit non-knapsack sub-additive relation (63) is identified by the *slack function* $\Delta(\xi) = \min_{i+j>n+1} \xi_{i+j-n-1} - \xi_i - \xi_j$.

Figure 5 continues to depict the same 3-dimensional cone on the left as shown in Figure 2. The red line segment on the right depicts $\Pi^\xi(K(6)) + 2lin(6)$. Note that π^+, π^0 and $\pi^- = \pi^{1-} = -\xi$ represent the same knapsack facet $\xi t \leq \xi_n = 1$ where π^{1-} in Figure 2 was

$$\pi^{1-} = \left(-0, -\frac{1}{3}, -\frac{1}{2}, -\frac{2}{3}, -1, -1\right).$$

The quadrilateral on the right in the figure is $\Pi(C_7, 6)$ contained in $S(K(6)) \cap \{\pi : \pi_n = 1\}$. The thick lines and the thin lines depict the knapsack sub-additive relations

$$\begin{aligned} 2\pi_1 &\geq \pi_2, \\ 3\pi_2 &\geq \pi_6 \qquad (\Leftrightarrow 2\pi_2 \geq \pi_4 \text{ corresponding to } 2+2=4), \end{aligned}$$

[6] This shooting at sub-additive relations here is not the shooting at knapsack facets discussed in the previous sections.

and the non-knapsack sub-additive relations

$$\pi_1 + 2\pi_2 \le 2 \qquad (\Leftrightarrow 2\pi_4 \ge \pi_1 \text{ corresponding to } 4+4 \equiv 1 \mod 7),$$
$$\pi_1 \le 3/4 \qquad (\Leftrightarrow \pi_3 + \pi_5 \ge \pi_1 \text{ corresponding to } 3+5 \equiv 1 \mod 7).$$

All points π on the edge from $lin(6)$ through π^+ represent the same knapsack facet $\xi t \le \xi_n = 1$ in $\pi t \ge \pi_n = 1$. In particular, π^ξ is the cyclic group facet representing the knapsack facet. In order to identify π^ξ, we shoot an arrow from $lin(n)$ at $\pi^+ = -\xi + 2lin(n)$ and see the point π^ξ where a non-knapsack relation is first hit.

The following theorem states a transformation by which our representation $\xi t \le 1$ of a knapsack facet can be converted into a cyclic group facet $\pi^\xi t \ge 1$ of $P(C_{n+1}, n)$.

Theorem 5.1 *Let* $\Delta(\xi) = \min_{i+j>n+1} \xi_{i+j-n-1} - \xi_i - \xi_j$ *and let*

$$\pi^\xi = \frac{n+1}{n \cdot \Delta(\xi) + (n+1)} \cdot (\xi - lin(n)) + lin(n).$$

Then, the knapsack facet $\xi t \le 1$ *with* $\xi_1 = 0$ *and* $\xi_n = 1$ *can be alternatively represented by* $\pi^\xi t \ge 1$ *which is a cyclic group facet for* $P(C_{n+1}, n)$.

Proof. The cyclic group facet π^ξ for $P(C_{n+1}, n)$ corresponding to a knapsack facet ξ for $P(K(n))$ is the point in $S(K(n)) \cap \{\pi : \pi_n = 1\}$ where shooting an arrow from $lin(n)$ at $\pi^+ = -\xi + 2lin(n)$ hits the first non-knapsack sub-additive relation as illustrated on the right in Figure 5. To perform the shooting, we translate π^+ and π^ξ to $\pi^0 = \pi^+ - lin(n) = -\xi + lin(n)$ and $\vec{\pi}^\xi = \pi^\xi - lin(n)$ in the cone $S(K(n)) \cap \{\pi : \pi_n = 0\}$. We see that π^0 spans an extreme ray of the cone $S(K(n)) \cap \{\pi : \pi_n = 0\}$ representing the knapsack facet $\xi t \le 1$ in $\pi^0 t \ge \pi^0_n = 0$. The translate $\vec{\pi}^\xi$ of π^ξ on the cone $S(K(n)) \cap \{\pi : \pi_n = 0\}$ is in the extreme ray spanned by π^0 and can be written as

$$\vec{\pi}^\xi = r\pi^0 \tag{64}$$

for some $r > 0$. We only need to show

$$r = \frac{n+1}{-n\Delta(\xi) - (n+1)} = \frac{\frac{n+1}{n}}{-\Delta(\xi) - \frac{n+1}{n}}$$

where $\Delta(\xi) = \min_{i+j>n+1} \xi_{i+j-n-1} - \xi_i - \xi_j$. Equivalently, we show

$$r \cdot \left(\min_{i+j>n+1} \{\xi_{i+j-n-1} - \xi_i - \xi_j\} + \frac{n+1}{n} \right) + \frac{n+1}{n} = 0. \tag{65}$$

The first non-knapsack sub-additive relation hit by the shooting is satisfied as equality, and π^ξ must satisfy

$$\pi^\xi_i + \pi^\xi_j \ge \pi^\xi_{i+j-n-1} \qquad \text{for every pair of } i \text{ and } j \text{ with } i+j > n+1,$$
$$\pi^\xi_i + \pi^\xi_j = \pi^\xi_{i+j-n-1} \qquad \text{for a pair of } i \text{ and } j \text{ with } i+j > n+1.$$

That is, π^ξ must satisfy

$$\min_{i+j>n+1} \left\{ \pi^\xi_i + \pi^\xi_j - \pi^\xi_{i+j-n-1} \right\} = 0. \tag{66}$$

It implies (65) as follows:

$$
\begin{aligned}
& r \cdot \left(\min_{i+j>n+1} \{\xi_{i+j-n-1} - \xi_i - \xi_j\} + \frac{n+1}{n} \right) + \frac{n+1}{n} \\
&= r \cdot \min_{i+j>n+1} \left\{ -\xi_i - \xi_j + \xi_{i+j-n-1} + \frac{n+1}{n} \right\} + \frac{n+1}{n} \\
&= r \cdot \min_{i+j>n+1} \left\{ \left(-\xi_i + \frac{i}{n}\right) + \left(-\xi_j + \frac{j}{n}\right) - \left(-\xi_{i+j-n-1} + \frac{i+j-n-1}{n}\right) \right\} + \frac{n+1}{n} \\
&= r \cdot \min_{i+j>n+1} \left\{ (-\xi + lin(n))_i + (-\xi + lin(n))_j - (-\xi + lin(n))_{i+j-n-1} \right\} + \frac{n+1}{n} \\
&= r \cdot \min_{i+j>n+1} \left\{ \pi^0_i + \pi^0_j - \pi^0_{i+j-n-1} \right\} + \frac{n+1}{n} \\
&= \min_{i+j>n+1} \left\{ \left(r\pi^0_i + \frac{i}{n}\right) + \left(r\pi^0_j + \frac{j}{n}\right) - \left(r\pi^0_{i+j-n-1} + \frac{i+j-n-1}{n}\right) \right\} \\
&= \min_{i+j>n+1} \left\{ \left(r\pi^0 + lin(n)\right)_i + \left(r\pi^0 + lin(n)\right)_j - \left(r\pi^0 + lin(n)\right)_{i+j-n-1} \right\} \\
&= \min_{i+j>n+1} \left\{ \pi^\xi_i + \pi^\xi_j - \pi^\xi_{i+j-n-1} \right\} = 0,
\end{aligned}
$$

completing the proof of the theorem. □

On the other hand, given a cyclic group facet $\pi t \geq \pi_n = 1$ for $P(C_{n+1}, n)$, the corresponding knapsack facet for $P(K(n))$ is alternatively represented in $\xi^\pi t \leq \xi^\pi_n = 1$ with $\xi^\pi_1 = 0$ as follows:

Theorem 5.2 *Let $\pi t \geq \pi_n = 1$ be a cyclic group facet adjacent to $lin(n)$ for $P(C_{n+1}, n)$. Then, it is a knapsack facet $\xi^\pi t \leq \xi^\pi_n = 1$ with $\xi^\pi_1 = 0$ for $P(K(n))$ given by*

$$\xi^\pi = -\frac{1}{n} \cdot \frac{\pi - lin(n)}{\pi_1 - \frac{1}{n}} + lin(n).$$

Its proof is similar to the proof of Theorem 5.1.

Figure 6 illustrates a facet represented in ξ and π interchangeably. In the figure, the coefficients of π are represented in blue and the coefficients of ξ are represented in red. For the knapsack facet illustrated in the figure, one representation can be transformed to the other using Theorems 5.1 and 5.2.

6. Conclusion

In this paper we exploit the fact that knapsack inequalities can be represented using either a sub-additive or super-additive characterization. In particular, we focus on $1/k$-inequalities for k dividing 6 or 8. We obtain a concise characterization of the super-additive version of these knapsack inequalities. This concise characterization allows us to efficiently separate these inequalities when solving knapsack problems in practice.

We use the sub-additive characterization of knapsack inequalities to develop a concise representation of the LP to be solved to identify the facet hit when performing the shooting experiment. This characterization allows us to significantly increase the size of problems over which we are able to perform the shooting experiment.

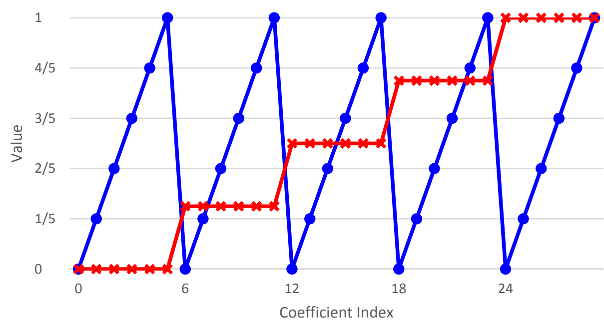

Figure 6: Coefficient vectors of a knapsack facet in ξ- and π-representations

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Appendix: The 1/8-facets Revisited

We prove Theorem 2.7 by enumerating all types of super-additive relations binding at $\xi^{8\text{-}(a_m)}$. Suppose that ξ is a solution to (5)-(8) that is binding at the same super-additive relations as $\xi^{8\text{-}(a_m)}$. If $\xi^{8\text{-}(a_m)}$ satisfies as equality a super-additive relation

$$\frac{1}{8}+\frac{1}{8}=\xi_i^{8\text{-}(a_m)}+\xi_j^{8\text{-}(a_m)}=\xi_{i+j}^{8\text{-}(a_m)}=\frac{2}{8},$$

those relations must include

$$\frac{1}{8}+\frac{1}{8}=\xi_{a_1}^{8\text{-}(a_m)}+\xi_{a_1}^{8\text{-}(a_m)}=\xi_{a_2}^{8\text{-}(a_m)}=\frac{2}{8}.$$

That is equivalent to Relation 1 of Theorem 2.7; *i.e.*,

$$2a_1 \le a_3 - 1.$$

Relation 1 through 5 of Theorem 2.7 include all possible cases as follows:

$$\begin{aligned}
&\frac{1}{8}+\frac{1}{8}=\xi_i^{8\text{-}(a_m)}+\xi_j^{8\text{-}(a_m)}=\xi_{i+j}^{8\text{-}(a_m)}=\frac{2}{8}\Leftrightarrow a_1+a_1\le a_3-1\Leftrightarrow \text{ Relation 1}\\
&\frac{1}{8}+\frac{2}{8}=\xi_i^{8\text{-}(a_m)}+\xi_j^{8\text{-}(a_m)}=\xi_{i+j}^{8\text{-}(a_m)}=\frac{3}{8}\Leftrightarrow a_1+a_2\le a_4-1\Leftrightarrow \text{ Relation 2}\\
&\frac{1}{8}+\frac{3}{8}=\frac{4}{8}\Leftrightarrow a_1+a_3\le a_5-1\Leftrightarrow a_1+a_3\le n+1-a_4-1=n-a_4\Leftrightarrow \text{ Relation 4}\\
&\frac{1}{8}+\frac{4}{8}=\frac{5}{8}\Leftrightarrow a_1+a_4\le a_6-1\Leftrightarrow a_1+a_4\le n+1-a_3-1=n-a_3\Leftrightarrow \text{ Relation 4}\\
&\frac{1}{8}+\frac{5}{8}=\frac{6}{8}\Leftrightarrow a_1+a_5\le a_7-1\Leftrightarrow a_1+n+1-a_4\le n+1-a_2-1\Leftrightarrow \text{ Relation 2}\\
&\frac{1}{8}+\frac{6}{8}=\frac{7}{8}\Leftrightarrow a_1+a_6\le a_8-1\Leftrightarrow a_1+n+1-a_3\le n+1-a_1-1\Leftrightarrow \text{ Relation 1}\\
&\frac{2}{8}+\frac{2}{8}=\frac{4}{8}\Leftrightarrow a_2+a_2\le a_5-1\Leftrightarrow 2a_2\le n+1-a_4-1=n-a_4\Leftrightarrow \text{ Relation 3}\\
&\frac{2}{8}+\frac{3}{8}=\frac{5}{8}\Leftrightarrow a_2+a_3\le a_6-1\Leftrightarrow a_2+a_3\le n+1-a_3-1=n-a_3\Leftrightarrow \text{ Relation 5}\\
&\frac{2}{8}+\frac{4}{8}=\frac{6}{8}\Leftrightarrow a_2+a_4\le a_7-1\Leftrightarrow a_2+a_4\le n+1-a_2-1=n-a_2\Leftrightarrow \text{ Relation 3}\\
&\frac{2}{8}+\frac{5}{8}=\frac{7}{8}\Leftrightarrow a_2+a_5\le a_8-1\Leftrightarrow a_2+n+1-a_4\le n+1-a_1-1\Leftrightarrow \text{ Relation 2}\\
&\frac{3}{8}+\frac{3}{8}=\frac{6}{8}\Leftrightarrow a_3+a_3\le a_7-1\Leftrightarrow a_3+a_3\le n+1-a_2-1=n-a_2\Leftrightarrow \text{ Relation 5}\\
&\frac{3}{8}+\frac{4}{8}=\frac{7}{8}\Leftrightarrow a_3+a_4\le a_8-1\Leftrightarrow a_3+a_4\le n+1-a_1-1=n-a_1\Leftrightarrow \text{ Relation 4.}
\end{aligned}$$

Recall that ξ is uniquely determined by three variables ξ_{a_1}, ξ_{a_2} and ξ_{a_3}, which are determined by any system of three (independent) equations from the following five equations equivalent to Relation 1 through Relation 5 of Theorem 2.7:

1. $\xi_{a_1}+\xi_{a_1}=\xi_{a_1+a_1}=\xi_{a_2}$
2. $\xi_{a_1}+\xi_{a_2}=\xi_{a_1+a_2}=\xi_{a_3}$

3. $\xi_{a_2} + \xi_{a_2} = \xi_{a_2+a_2} = 1/2$
4. $\xi_{a_1} + \xi_{a_3} = \xi_{a_1+a_3} = 1/2$
5. $\xi_{a_2} + \xi_{a_3} = \xi_{a_5} = 1 - \xi_{a_3}$

$\square$