MAXIMAL VERTEX-CONNECTIVITY OF $\overrightarrow{S_{n,k}}$

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Abstract. The class of star graphs is a popular topology for interconnection networks. However it has certain deficiencies. A class of generalization of star graphs called (n, k)-star graphs was introduced in [12] to address these issues. In this paper we will consider the vertex-connectivity of the directed (n, k)-star graph, $\overrightarrow{S_{n,k}}$, given in [8], and show that it is maximally connected.

1. Introduction

Directed interconnection networks have gained much attention in the area of distributed computing. Recent research in this area includes [5, 11, 15, 16, 20, 22]. The study of using unidirectional hypercubes as the basis for high speed networking can be found in [15]. For a more general model, we refer the reader to [11] for an architectural model for the studies of unidirectional graph topologies and a specific application, which also includes a comparison of the diameters among some unidirectional interconnection networks.

One of the most popular interconnection networks is the star graph, S_n, proposed in [24]. It was introduced as a competitive model to the hypercube, Q_n. It has many advantages over the hypercube including lower degree and a smaller diameter. Day and Tripathi proposed an orientation of the star graph in [20]. They gave an efficient near-optimal distributed routing algorithm for it. One of the main criteria of a good interconnection network topology is connectivity. The ideal situation is for a unidirectional graph topology to have the highest possible connectivity. Indeed, Jwo and Tuan [22] showed that the unidirectional hypercube proposed by Chou and Du [15] has this important property. Since the star graph was introduced as a competitive alternative to the hypercube, it is necessary that an orientation for the star graph has that same property for it to remain competitive. Indeed, [4] studied the arc-connectivity of this graph. Later [6] showed that it has the highest possible vertex-connectivity.

Although S_n has proven to be an attractive alternative to Q_n, one drawback it has is the restriction on the number of vertices. (Q_n also has this drawback though not as severe.) Since S_n has $n!$ vertices, anyone wanting to build a multiprocessor network using this topology is forced to build one with $n!$ vertices for some value of n. This led in part to the introduction of (n, k)-star graphs in [12], which is a generalization of star graphs. This graph is denoted by $S_{n,k}$. In [8] an orientation of these graphs is proposed and their properties including arc-connectivity, diameter as well as distributed routing algorithms are studied. In this paper, we show that they have the highest possible vertex-connectivity.

2. Preliminaries

Some recent papers on star graphs or generalizations of star graphs include [1–10, 12–14, 17–21, 23–26]. Basic terminology in graph theory can be found in [27]. Given a directed graph $\overrightarrow{D}$, $\overleftarrow{D}$ denotes the graph obtained from $\overrightarrow{D}$ by reversing directions on all arcs. An (n, k)-star graph $S_{n,k}$ with $1 \leq k < n$ is governed by the two parameters n and k. The vertex-set of $S_{n,k}$ consists of all the permutations of k elements chosen from the ground set $\{1, 2, \ldots, n\}$. Two vertices $[a_1, a_2, \ldots, a_k]$ and $[b_1, b_2, \ldots, b_k]$ are adjacent if one of the following holds:

(1) There exists a $2 \leq r \leq k$ such that $a_1 = b_r$, $a_r = b_1$ and $a_i = b_i$ for $i \in \{1, 2, \ldots, k\} \setminus \{1, r\}$.

(2) $a_i = b_i$ for $i \in \{2, \ldots, k\}$, $a_1 \neq b_1$.

Hence given a vertex $[a_1, a_2, \ldots, a_k]$, it has $k - 1$ neighbours via the adjacency rule (1) by exchanging a_1 with each of a_i, $i \in \{2, 3, \ldots, k\}$, and it has $n - k$ neighbours via the adjacency rule (2) by exchanging a_1 with each element in $\{1, 2, \ldots, n\} \setminus \{a_1, a_2, a_3, \ldots, a_k\}$. We note that adjacency rule (1) is precisely the rule for star graphs. In keeping with the terminology for star graphs, an edge corresponding to this rule is a *star-edge*; it will be called an *i-edge* if the exchange is between position 1 and position i where $i \in \{2, 3, \ldots, k\}$. An edge corresponding to the second rule is a *residual-edge*. Figure 1 gives $S_{4,2}$. (We note that for convenience, we

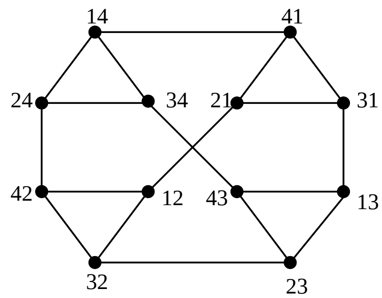

FIGURE 1. $S_{4,2}$

write the (n,k)-permutation $[i,j]$ as ij, for example, $[1,4]$ as 14.) Note that given an edge in $S_{n,k}$ with the labellings of its two end-vertices, one can immediately determine whether it is a star-edge or a residual-edge. The family of $S_{n,k}$ generalizes the star graph, as $S_{n,n-1}$ is isomorphic to the star graph S_n. For $S_{n,n-1}$, the unique residual-edge for each vertex can be viewed as an *n-edge* in S_n. Since the graph reduces to the complete graph if $k=1$, we assume $k \geq 2$ for the rest of the paper.

The next result contains some elementary properties of (n,k)-star graphs whose proofs are obvious. Additional properties can be found in [12].

Proposition 2.1. *The (n,k)-star graph $S_{n,k}$ has $n!/(n-k)!$ vertices and is a regular graph with degree $n-1$. Moreover,*

(1) *Let $\{x_1, x_2, \ldots, x_k\} \subseteq \{1,2,\ldots,n\}$ with $k \geq 3$. Let G be the subgraph of $S_{n,k}$ induced by vertices whose labellings are permutations of $x_1, x_2, \ldots, x_k$. Then G is isomorphic to the star graph S_k.*

(2) *Let $\{x_2, x_3, \ldots, x_k\} \subseteq \{1,2,\ldots,n\}$. Let G be the subgraph of $S_{n,k}$ induced by vertices of the form $[y_1, x_2, x_3, \ldots, x_k]$ where $y_1 \in \{1,2,\ldots,n\} \setminus \{x_2, x_3, \ldots, x_k\}$. Then G is isomorphic to K_{n-k+1}, the complete graph on $n-k+1$ vertices.*

(3) *Let G be a subgraph of $S_{n,k}$ with $k \geq 3$ induced by vertices with labellings having the same symbol in the rth position where $2 \leq r \leq k$. Then G is isomorphic to $S_{n-1,k-1}$.*

A star subgraph of $S_{n,k}$ using the rule in (1) of Theorem 2.1 will be called a *fundamental star*. If $k = 2$, then the subgraph via (1) of Theorem 2.1 is K_2, which is not a star graph. However, we will still refer to it as a fundamental star. (The star graph S_n was defined for $n \geq 3$. If one backwardly extends the definition to the case $n = 2$, then "S_2" is indeed K_2.) A complete subgraph of $S_{n,k}$ using the rule in (2) of Theorem 2.1 will be called a *fundamental clique*. It is clear that there are $\binom{n}{k}$ fundamental stars and $\binom{n}{k-1}(k-1)! = \frac{n!}{(n-k+1)!}$ fundamental cliques.

3. Unidirectional (n,k)-star graphs

A directed graph D is said to be *maximally connected* if it is k-connected with $k = \min_{v \in V(D)}\{\delta(v), \rho(v)\}$ where $\delta(v)$ ($\rho(v)$) is the out-degree (in-degree) of v. In [8], an orientation for $S_{n,k}$ is introduced together with some of its properties. We will show that this directed graph, denoted by $\overrightarrow{S_{n,k}}$, is *maximally connected*. This result was proven to be correct for the case $\overrightarrow{S_{n,n-1}}$ (in the language of star graphs) in [6].

A directed star-edge is a *star-arc*. A directed residual-edge is a *residual-arc*. Now associate each vertex $[a_1, a_2, \ldots, a_k]$ in $\overrightarrow{S_{n,k}}$, with the following *associated permutation* on n symbols $[a_1, a_2, \ldots, a_k, x_1, \ldots, x_{n-k}]$ where $x_1 < x_2 < \ldots < x_{n-k}$. A vertex in $\overrightarrow{S_{n,k}}$ is *even* (*odd*) if its associated permutation is even (odd). It is easy to see that each vertex belongs to exactly one fundamental star and one fundamental clique. Suppose an r-regular graph with odd vertices and even vertices is given. If r is even, then an assigned orientation is *balanced* if the resulting directed graph is $\frac{r}{2}$-regular. If r is odd, then an assigned orientation is *odd-more-out* if (1) the out-degree is $\lceil \frac{r}{2} \rceil$ and the in-degree is $\lfloor \frac{r}{2} \rfloor$ for every odd vertex in the resulting graph, and (2) the in-degree is $\lceil \frac{r}{2} \rceil$ and the out-degree is $\lfloor \frac{r}{2} \rfloor$ for every even vertex in the resulting graph. (The term "odd-more-out" indicates an odd vertex has a higher out-degree than in-degree.) The term *odd-more-in* is defined similarly. The orientation rule given in [8] satisfies the objective stated in Table 1. This will give an odd (even) vertex with in-degree (out-degree) $\lceil \frac{n-1}{2} \rceil$ and out-degree (in-degree) $\lfloor \frac{n-1}{2} \rfloor$, that is, balanced if n is odd and odd-more-in if n is even.

TABLE 1. Orientation objective

k	$n-k+1$	fundamental star	fundamental clique	
even	even	odd-more-in	odd-more-out	Figure 2 (basic)
even	odd	odd-more-in	balanced	Figure 4 (basic)
odd	even	balanced	odd-more-in	Figure 3 (reverse)
odd	odd	balanced	balanced	Figure 4 (basic)

We will now give the orientation of $S_{n,k}$. As mentioned earlier, it can be locally determined if an edge in $S_{n,k}$ is a star-edge or a residual-edge. For the star-edges it is easy to see that the end-vertices of a star-edge are of opposite parity. Suppose $\pi_a\pi_b$ is a star-edge, where π_a and π_b are vertices, without loss of generality we may assume π_a is even and π_b is odd. We use the standard Day-Tripathi rule for star graphs: if $\pi_a\pi_b$ is an i-edge then the arc is oriented from π_a to π_b when i is even and from π_b to π_a when i is odd. This accomplishes the orientation objective for the fundamental stars given in Table 1.

To orient residual-edges, we do the following. For any residual-edge it is easy to see which fundamental clique it belongs to. The vertices will be of the form $\pi_i = [x_i, a_2, \ldots, a_k]$ and $\pi_j = [x_j, a_2, \ldots, a_k]$, and they belong to the fundamental clique with vertices $\pi_1 = [x_1, a_2, \ldots, a_k], \pi_2 = [x_2, a_2, \ldots, a_k], \ldots, \pi_{n-k+1} = [x_{n-k+1}, a_2, \ldots, a_k]$ where $x_1 < x_2 < \ldots < x_{n-k+1}$. This is the *natural ordering* of the vertices in this fundamental clique and π_1 is its *leading vertex*. It is clear that if we consider the vertices in this order, the parity of their associated permutations alternates. In [8], the following easy result is used to orient the fundamental cliques.

Proposition 3.1. *Let $v_1, v_2, \ldots, v_m$ be vertices of K_m with $m \geq 2$ where v_i is odd if and only if i is odd. Suppose $i < j$. Then orient the edge from v_i to v_j if they have different parity and orient the edge from v_j to v_i if they have the same parity. Then the resulting directed graph has an odd-more-out orientation if m is even and balanced orientation if m is odd.*

Proposition 3.1 suggests the following rule for the case with k even and $n-k+1$ even. *Basic rule:* If the leading vertex π_1 is odd, orient the edges as follows: Given two vertices π_i and π_j where $x_i < x_j$, orient the edge from π_i to π_j if π_i and π_j have different parity, and from π_j to π_i if π_i and π_j have the same parity. If the leading vertex is even, the orientations are reversed. See Figure 2 for an example. For the case with k odd and $n-k+1$ even, we

FIGURE 2. Basic rule for the natural ordering from left to right

reverse the direction resulting from the basic rule and it is the *reverse rule*. See Figure 3 for an example. If $n-k+1$ is odd, [8] observed that the orientation resulting from the basic

FIGURE 3. Reverse rule for the natural ordering from left to right

rule and the reverse rule give a balanced orientation. The results in [8] are independent of the precise orientation. In this paper, it is more natural to use the basic rule. See Figure 4 for an example.

FIGURE 4. The natural ordering from left to right

One can check that $\overrightarrow{S_{n,n-1}}$ is isomorphic to the orientation of the star graph S_n given in [20]. Additional properties about this orientation and justification behind it can be found in [8].

Since the orientation objective in Table 1 is achieved and each vertex belongs to exactly one fundamental star and one fundamental clique, we have the following proposition.

Proposition 3.2. [1] *Let $k \geq 2$. If $n-1$ is even then $\overrightarrow{S_{n,k}}$ is $\frac{n-1}{2}$-regular. If $n-1$ is odd then every odd (even) vertex in $\overrightarrow{S_{n,k}}$ has in-degree (out-degree) $\lceil \frac{n-1}{2} \rceil$ and out-degree (in-degree) $\lfloor \frac{n-1}{2} \rfloor$.*

4. Maximal Connectivity

We first consider the connectivity result for the directed complete graph. Define $\overrightarrow{K_n}$ by orienting K_n on $\{1, 2, \ldots, n\}$ as follows: For vertices u and v with $u > v$, direct the arc from u to v if u and v are of different parity, and from v to u if they are of the same parity. (Note that the orientation given here is the opposite of Proposition 3.1. The reason for this is to make the statement of Lemma 4.5 cleaner.)

Lemma 4.1. *Let $2p+1 \geq 3$. Then $\overrightarrow{K_{2p+1}}$ is p-connected.*

Proof. We will show this by induction. Clearly $\overrightarrow{K_3}$ is 1-connected and $\overrightarrow{K_5}$ is 2-connected. Partition the vertices of $\overrightarrow{K_{2p+1}}$ as follows: Place $2p$ and $2p+1$ in their own sets and let A be all the odd indexed vertices besides $2p+1$, let B be all the even indexed vertices besides $2p$, as pictured in Figure 5.

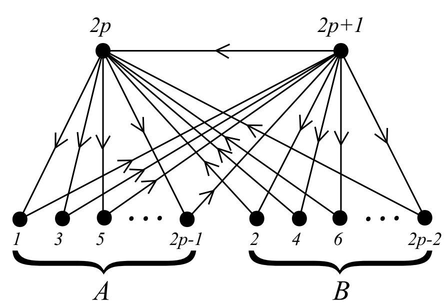

Figure 5. Graph from Lemma 4.1

[1]Since the routing result in [8] is designed for $k \geq 3$, some results including this one was stated for $k \geq 3$.

Consider deleting S from $\overrightarrow{K_{2p+1}}$ where $|S| = p-1$.

Case 1: $2p$, $2p+1 \notin S$

Since $|S| = p-1$ and $|A| = p$, the digraph X induced by $2p$, $2p+1$, and $A \setminus S$ is strongly connected, as there is a directed arc from $2p+1$ to $2p$, and every vertex in A has an arc leaving it directed into $2p+1$ and an arc entering it from $2p$. Now for every $u \in B \setminus S$, there is an arc from u to X and vice versa. Hence $\overrightarrow{K_{2p+1}} \setminus S$ is strongly connected.

Case 2: $2p$, $2p+1 \in S$

Since $\overrightarrow{K_{2p+1}} \setminus \{2p, 2p+1\}$ is isomorphic to $\overrightarrow{K_{2p-1}}$ which is $(p-1)$-connected by induction, deleting $S \setminus \{2p, 2p+1\}$ from it will not disconnect it since $|S \setminus \{2p, 2p+1\}| = p-3$. Therefore $\overrightarrow{K_{2p+1}} \setminus S$ is strongly connected.

Case 3: $2p \in S$, $2p+1 \notin S$

By our induction step, $\overrightarrow{K_{2p-1}}$ is $(p-1)$-connected. Since $|S \setminus \{2p\}| = p-2$ we know that $\overrightarrow{K_{2p-1}} \setminus (S \setminus \{2p\})$ must be strongly connected. Since $|A| = p$ and $|B| = p-1$ and $|S \setminus \{2p\}| = p-2$, there must be vertices from both A and B that are not in S. Since every vertex in B has an arc directed to it from $2p+1$ and every vertex in A has an arc directed from it to $2p+1$, $\overrightarrow{K_{2p+1}} \setminus S$ is strongly connected.

Case 4: $2p \notin S$, $2p+1 \in S$

Similar to case 3.

Therefore, $\overrightarrow{K_{2p+1}}$ is p-connected. □

Lemma 4.2. *Let $2p \geq 2$. Then $\overrightarrow{K_{2p}}$ is $(p-1)$-connected.*

Proof. Since $\overrightarrow{K_{2p}}$ is isomorphic to $\overrightarrow{K_{2p+1}} \setminus \{2p+1\}$, the result follows from Lemma 4.1. (Note that if $p = 1$, the result is trivially true.) □

Proposition 4.3. *Let F be a fundamental clique in $\overrightarrow{S_{n,k}}$. Then F is isomorphic to either $\overrightarrow{K_{n-k+1}}$ or $\overleftarrow{K_{n-k+1}}$ and hence $\left\lfloor \frac{n-k}{2} \right\rfloor$-connected.*

Proof. Since F has $n-k+1 \geq 2$ vertices, we may apply Lemma 4.1 and Lemma 4.2. Let $v_1, v_2, \ldots, v_{n-k+1}$ be vertices of F in its natural ordering. Then by the definition of $\overrightarrow{S_{n,k}}$, the mapping ϕ from $V(F)$ to $\{1, 2, \ldots, n-k+1\}$ defined by $\phi(v_i) = i$ is an isomorphism from F to either $\overrightarrow{K_{n-k+1}}$ or $\overleftarrow{K_{n-k+1}}$. □

Lemma 4.4. *Suppose $n \geq 3$. Let H be the graph obtained from $S_{n,2}$ by contracting each fundamental clique to a vertex. Then the resulting graph is isomorphic to K_n.*

Proof. Since there is exactly one edge between every pair of fundamental cliques, the result follows. □

Lemma 4.5. *Let $\overrightarrow{H}$ be the directed graph obtained from $\overrightarrow{S_{n,2}}$ by contracting each fundamental clique to a vertex, then the resulting directed graph is isomorphic to $\overrightarrow{K_n}$ and hence $\lfloor \frac{n-1}{2} \rfloor$-connected.*

Proof. Since $k = 2$, each fundamental clique is uniquely identified by the symbol in the second position of its vertices. Let F_i represent the fundamental clique where i is located in the second position, that is, vertices in F_i are of the form $[x, i]$ where $x \in \{1, 2, \ldots, n\} \setminus \{i\}$. Now for each distinct F_i and F_j there is exactly one arc between them, which is between vertices $[j, i]$ and $[i, j]$. Hence we labelled the vertices of $\overrightarrow{H}$ by the fundamental cliques of $\overrightarrow{S_{n,2}}$, namely, $F_1, F_2, \ldots, F_n$. To show that $\overrightarrow{H}$ is is isomorphic to $\overrightarrow{K_n}$, we map F_i in $\overrightarrow{H}$ to i in $\overrightarrow{K_n}$. We will show that this is an isomorphism. Note that in the next claim, the parity of a vertex is the parity of a permutation whereas the parity of a number is its numeric parity; nevertheless, each one is either even or odd.

Claim: The leading vertex in F_i has the same parity as i.

Justification: For $i = 1$ it is clear that the leading vertex, $[2, 1]$, whose associated permutation $[2, 1, 3, 4, \ldots, n]$ is odd. For $i \neq 1$, the leading vertex, $[1, i]$ shares the same parity as i because the number of transpositions required to turn its associated permutation to the identity permutation, is 0 if $i = 2$, and $i - 2$ otherwise. ◇

Claim:

(1) Suppose p is odd. Then $[q,p]$ in F_p and q have the same parity if $q < p$, and they have opposite parity if $q > p$.

(2) Suppose p is even. Then $[q,p]$ in F_p and q have opposite parity if $q < p$, and they have the same parity if $q > p$.

Justification: The vertices of F_1 are $[2,1],[3,1],\ldots,[n,1]$, the vertices of F_i where $1 < i < n$ are $[1,i],[2,i],\ldots,[i-1,i],[i+1,i],\ldots,[n,i]$, and the vertices of F_n are $[1,n],[2,n],\ldots,[n-1,n]$. In every case, the associated permutation of each vertex in the list (other than the first one) can be obtained from the associated permutation of the previous vertex through a multiplication of a single transposition. It is now easy to see that the statement is true by combining this observation and the previous claim. $\diamond$

Now consider different cases regarding the parity of F_i and F_j, that is, the parity of i and j. We will consider the direction of the unique arc between F_i and F_j. Note that this is a star-arc, in fact, it is an oriented 2-edge as $n = 2$. So following the orientation rule, it is directed from an even permutation to an odd permutation. Without loss of generality, assume $i < j$.

Case 1: i is odd and j is even.

$[j,i]$ has the opposite parity as j and $[i,j]$ has the opposite parity as i, so $[j,i]$ is odd and $[i,j]$ is even. Therefore the arc will be directed from $[i,j]$ to $[j,i]$. So the unique arc goes from F_j to F_i as required.

Case 2: i is even and j is odd.

In this case, since $i < j$, $[j,i]$ will be odd and $[i,j]$ will be even. So the arc is directed from $[i,j]$ to $[j,i]$. Hence the unique arc is from F_j to F_i as required.

Case 3: Both i and j are even.

In this case, $[j,i]$ will have the same parity as j and therefore even. Vertex $[i,j]$ will have opposite parity as i and will be odd. Therefore the arc is directed from $[j,i]$ to $[i,j]$. So the unique arc goes from F_i to F_j as required.

Case 4: Both i and j are odd.

In this case, $[j,i]$ will have the opposite parity as j and is therefore even. Vertex $[i,j]$

will have the same parity as i and is therefore odd. This means the arc is from $[j, i]$ to $[i, j]$. So the unique arc goes from F_i to F_j as required.

Hence the mapping from F_i in $\overrightarrow{H}$ to i in $\overrightarrow{K_n}$ is indeed an isomorphism, and therefore $\overrightarrow{H}$ is $\lfloor \frac{n-1}{2} \rfloor$-connected by Lemma 4.1 and Lemma 4.2. □

Theorem 4.6. *$\overrightarrow{S_{2p+1,2}}$ is p-connected when $2p+1 \geq 3$.*

Proof. Clearly $\overrightarrow{S_{3,2}}$ is strongly connected. We may assume $2p+1 \geq 5$. Let us delete a subset of the vertices S from $\overrightarrow{S_{2p+1,2}}$ with $|S| = p-1$. We will call a fundamental clique damaged if it contains elements from S; otherwise it is undamaged. Let d be the number of damaged fundamental cliques. Since $|S| = p-1$ we know that $d \leq p-1$. Let X be the digraph induced by the vertices of the undamaged fundamental cliques. Since each fundamental clique is strongly connected it follows from Lemma 4.5 that X is strongly connected.

Case 1: Every damaged fundamental clique has at most $p-2$ vertices from S.

In this case, every damaged fundamental clique is strongly connected by Proposition 4.3. Choose any damaged fundamental clique D with α deleted vertices. Since it is strongly connected, it is enough to find a directed arc from it to X and vice versa. There are p arcs leaving D, each to a different fundamental clique. Since D has α deleted vertices, there are at most $p-1-\alpha$ additional damaged fundamental cliques. Hence there are at least $\alpha+1$ undamaged fundamental cliques among the ones those p arcs go to. Thus there is at least one arc from D to X. Similarly, there is at least one arc from X to D. So $\overrightarrow{S_{2p+1,2}} \setminus S$ is strongly connected.

Case 2: There exists one damaged fundamental clique, A, with $p-1$ deleted vertices.

Then A is the unique damaged fundamental clique. Let Y denote the strong component of $\overrightarrow{S_{2p+1,2}} \setminus S$ containing X. Suppose u is a vertex in A that is not deleted. Now suppose the $2p$ vertices in A, when listed in their natural order are odd, even,. . ., odd, even. Now consider two cases. (Note that every arc from A to X is a star-arc, in fact, it is an oriented 2-edge as $n = 2$. So following the rule, it is directed from an even permutation to an odd permutation.)

Case 2a: u is even.

Note that since k and $n-k+1$ are even, the basic rule applies in orienting the fundamental cliques. Since u is even, there is an arc from u to a vertex in X. In order to show that u is part of Y, we need only show that there exists a vertex in X with a directed path from it to u. Construct sets C_i, with $1 \le i \le p$, by assigning each vertex with an odd index that is less than u, and each consecutive pair of vertices with index larger than u in the order *odd-even*, to one of the C_i's, as pictured in Figure 6.

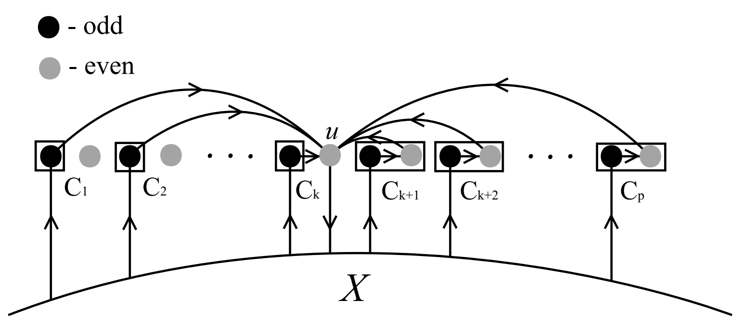

FIGURE 6. Picture from Case 2a

Since $|A| = 2p$, it is clear that there there are in fact p such sets. It is clear that this gives p mutually (internal) disjoint directed paths from X to u. (See Figure 6.) Since we only delete $p-1$ vertices in total, at least one of these paths remains intact, and thus u is part of Y.

Case 2b: u is odd.

Note that since k and $n-k+1$ are even, the basic rule applies in orienting the fundamental cliques. Since u is odd, there is an edge entering u from X. In order to show that u is part of Y, we need only show that there exists a vertex in X with a directed path from u to X. Construct sets C_i, with $1 \le i \le p$, by assigning each vertex with an even index that is greater than u, and each consecutive pair of vertices with lower index than u in order *odd-even*, to one of the C_i's, as pictured in Figure 7.

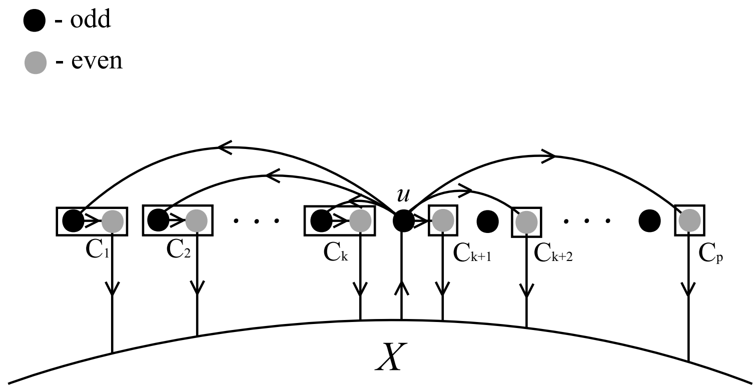

FIGURE 7. Picture from Case 2b

Since $|A| = 2p$, it is clear that there there are in fact p such sets, and hence there are p mutually (internal) disjoint directed paths from u to to X. (See Figure 7.) Since we only delete $p-1$ vertices, at least one of these paths remains intact, and thus u is part of Y.

Similarly, the result is true if the $2p$ vertices in A whose natural order is of the form even, odd,. . ., even, odd.

□

Theorem 4.7. $\overrightarrow{S_{2p,2}}$ *is* $(p-1)$*-connected when* $2p \geq 4$.

Proof. It is clear that $\overrightarrow{S_{4,2}}$ is strongly connected. We may assume $2p \geq 6$. In $\overrightarrow{S_{2p,2}}$ there are $2p$ fundamental cliques, each of size $2p-1$, each of which is $(p-1)$-connected by Proposition 4.3. Let us consider deleting S from $\overrightarrow{S_{2p,2}}$ where $|S| = p-2$. Define a damaged fundamental clique as a fundamental clique containing elements of S, and an undamaged fundamental clique as one that is not damaged. Let d be the number of damaged fundamental cliques. Since $|S| = p-2$ we know that $d \leq p-2$. Let X be the digraph induced by the vertices in the undamaged fundamental cliques. Since each fundamental clique is strongly connected, it follows from Lemma 4.5 that X is strongly connected. Since $|S| = p-2$, every damaged fundamental clique has at most $p-2$ deleted vertices, and therefore remains strongly connected when these vertices are deleted by Proposition 4.3. Choose any damaged fundamental clique D with α deleted vertices. Since it is strongly connected, it is enough to find a directed arc from it to X and vice versa. There are at least $p-1$ arcs leaving D, each to a different fundamental clique. Since D has α deleted vertices, there are at most $p-2-\alpha$ additional damaged fundamental cliques. Hence there are at least $\alpha+1$ undamaged fundamental cliques among the ones those $p-1$ arcs go to. Thus there is at least one arc from D to X. Similarly, there is at least one arc from X to D. So $\overrightarrow{S_{2p,2}} \setminus S$ is strongly connected. □

Corollary 4.8. *Let* $n \geq 3$. $\overrightarrow{S_{n,2}}$ *is* $\lfloor \frac{n-1}{2} \rfloor$*-connected.*

Proof. This follows directly from the previous two theorems. □

Lemma 4.9. *Let* $a \in \{1, 2, \ldots, n\}$ *and* $r \in \{2, 3, \ldots, k\}$. *Let* Q *be the subgraph of* $\overrightarrow{S_{n,k}}$ *induced by vertices with the symbol* a *in the* r*th position. Let* ϕ *be the function that maps* $[a_1, a_2, \ldots, a_{r-1}, a, a_{r+1}, \ldots, a_k]$ *on* $\{1, 2, \ldots, n\}$ *to* $[a_1, a_2, \ldots, a_{r-1}, a_{r+1}, \ldots, a_k]$ *on* $\{1, 2, \ldots, n\} \setminus \{a\}$. *Then* ϕ *either uniformly preserves the parity or uniformly reverses the parity of the vertices.*

Proof. We consider $\overrightarrow{S_{n-1,k-1}}$ on the symbol-set $\{1, 2, \ldots, n\} \setminus \{a\}$. For notational simplicity, we may assume $r = k$. Consider the mapping ϕ from the vertex-set of Q to the vertex-set of $\overrightarrow{S_{n-1,k-1}}$ defined by $\phi([x_1, x_2, \ldots, x_{k-1}, a]) = [x_1, x_2, \ldots, x_{k-1}]$. We claim that $[x_1, x_2, \ldots, x_{k-1}, a]$ and $[x_1, x_2, \ldots, x_{k-1}]$ have either the same parity for every $[x_1, x_2, \ldots, x_{k-1}]$ or have opposite parity for every $[x_1, x_2, \ldots, x_{k-1}]$. Consider $[x_1, x_2, \ldots, x_{k-1}, a]$. Its associated permutation is $u = [x_1, x_2, \ldots, x_{k-1}, a, t_{k+1}, \ldots, t_n]$ with $t_{k+1} < t_{k+2} < \cdots < t_n$. Let π be the permutation on $\{1, 2, \ldots, n\}$ obtained by exchanging the symbols a and k from the identity permutation. To compute the parity of u, we compute the number of transpositions required from u to π plus 1. To compute the parity of $u_1 = [x_1, x_2, \ldots, x_{k-1}, t_{k+1}, \ldots, t_n]$, let π_1 be the permutation on $\{1, 2, \ldots, n\} \setminus \{a\}$ obtained by deleting a from π. We note that a is in the kth position in π. Then the parity of u_1 is the parity of the number of transpositions required from u_1 to π_1 and from π_1 to the identity permutation on $\{1, 2, \ldots, n\} \setminus \{a\}$. Since the number of transpositions required from u to π is the same as the number of transpositions required from u_1 to π_1, the claim is established as the the number of transpositions required from π and π_1 to their respective identities is independent of u and u_1. □

Lemma 4.10. *Let $k \geq 3$, $n-k+1$ be even, and $a \in \{1, 2, \ldots, n\}$. Let Q be subgraph of $\overrightarrow{S_{n,k}}$ induced by vertices with a in the 2nd position. Then Q has the same connectivity as $\overrightarrow{S_{n-1,k-1}}$.*

Proof. We first assume k is even. Consider the mapping ϕ from the vertex-set of Q to the vertex-set of $\overrightarrow{S_{n-1,k-1}}$ defined by $\phi([x_1, a, x_3, \ldots, x_k]) = [x_1, x_3, \ldots, x_k]$. Suppose $[x_1, a, x_3, \ldots, x_k]$ is even. The case $[x_1, a, x_3, \ldots, x_k]$ being odd is similar and will be omitted. By Lemma 4.9, ϕ either uniformly preserves or reverses parity of vertices in Q. We consider two subcases.

The first case is ϕ preserves parity, that is, $[x_1, x_3, \ldots, x_k]$ is even. Then the neighbours of $[x_1, a, x_3, \ldots, x_k]$ in Q via star-arcs are

$$
\begin{gathered}
[x_1, a, x_3, \ldots, x_k] \longleftarrow [x_3, a, x_1, x_4, \ldots, x_k], \\
[x_1, a, x_3, \ldots, x_k] \longrightarrow [x_4, a, x_3, x_1, \ldots, x_k], \\
[x_1, a, x_3, \ldots, x_k] \longleftarrow [x_5, a, x_3, x_4, x_1, \ldots, x_k], \\
\vdots \\
[x_1, a, x_3, \ldots, x_k] \longrightarrow [x_k, a, x_3, x_4, \ldots, x_{k-1}, x_1].
\end{gathered}
$$

However, in $\overrightarrow{S_{n-1,k-1}}$ on the symbols $\{1, 2, \ldots, n\} \setminus \{a\}$, we have

$$
\begin{gathered}
[x_1, x_3, \ldots, x_k] \longrightarrow [x_3, x_1, x_4, \ldots, x_k], \\
[x_1, x_3, \ldots, x_k] \longleftarrow [x_4, x_3, x_1, \ldots, x_k], \\
[x_1, x_3, \ldots, x_k] \longrightarrow [x_5, x_3, x_4, x_1, \ldots, x_k], \\
\vdots \\
[x_1, x_3, \ldots, x_k] \longleftarrow [x_k, x_3, x_4, \ldots, x_{k-1}, x_1].
\end{gathered}
$$

Now for the residual-arcs, care must be taken. The basic rule is used in Q but the reverse rule is used in $\overrightarrow{S_{n-1,k-1}}$. We note that ϕ preserves the natural ordering of a fundamental clique. In summary, ϕ preserves parity, residual-arcs in Q are oriented by basic rule and residual-arcs in $\overrightarrow{S_{n-1,k-1}}$ use reverse rule. Therefore, ϕ reverses directions of the residual-arcs. (Compare the left picture in Figure 2 and the left picture in Figure 3.) Hence Q is isomorphic to $\overleftarrow{S_{n-1,k-1}}$ via ϕ.

The second case is ϕ reverses parity, that is, $[x_1, x_3, \ldots, x_k]$ is odd. Then the neighbours of $[x_1, a, x_3, \ldots, x_k]$ in Q via star-arcs are

$$\begin{gathered}
[x_1, a, x_3, \ldots, x_k] \longleftarrow [x_3, a, x_1, x_4, \ldots, x_k], \\
[x_1, a, x_3, \ldots, x_k] \longrightarrow [x_4, a, x_3, x_1, \ldots, x_k], \\
[x_1, a, x_3, \ldots, x_k] \longleftarrow [x_5, a, x_3, x_4, x_1, \ldots, x_k], \\
\cdots \\
[x_1, a, x_3, \ldots, x_k] \longrightarrow [x_k, a, x_3, x_4, \ldots, x_{k-1}, x_1].
\end{gathered}$$

However, in $\overrightarrow{S_{n-1,k-1}}$ on the symbols $\{1, 2, \ldots, n\} \setminus \{a\}$, we have

$$\begin{gathered}
[x_1, x_3, \ldots, x_k] \longleftarrow [x_3, x_1, x_4, \ldots, x_k], \\
[x_1, x_3, \ldots, x_k] \longrightarrow [x_4, x_3, x_1, \ldots, x_k], \\
[x_1, x_3, \ldots, x_k] \longleftarrow [x_5, x_3, x_4, x_1, \ldots, x_k], \\
\vdots \\
[x_1, x_3, \ldots, x_k] \longrightarrow [x_k, x_3, x_4, \ldots, x_{k-1}, x_1].
\end{gathered}$$

Now for the residual-arcs, care must be taken. The basic rule is used in Q but the reverse rule is used in $\overrightarrow{S_{n-1,k-1}}$. We note that ϕ preserves the natural ordering of a fundamental clique. In summary, ϕ reverses parity, residual-arcs in Q are oriented by basic rule and residual-arcs in $\overrightarrow{S_{n-1,k-1}}$ use reverse rule. Therefore ϕ preserves directions of the residual-arcs. (The double reversal of orientations preserves the directions. Compare the left picture in Figure 2 and the right picture in Figure 3.) Hence Q is isomorphic to $\overrightarrow{S_{n-1,k-1}}$ via ϕ.

The case k being odd is similar. $\square$

Lemma 4.11. *Let $k \geq 3$, $n - k + 1$ be odd, and $a \in \{1, 2, \ldots, n\}$. Let Q be subgraph of $\overrightarrow{S_{n,k}}$ induced by vertices with a in the kth position. Then Q has the same connectivity as $\overrightarrow{S_{n-1,k-1}}$.*

Proof. We first assume k is even. Consider the mapping ϕ from the vertex-set of Q to the vertex-set of $\overrightarrow{S_{n-1,k-1}}$ defined by $\phi([x_1, x_2, \ldots, x_{k-1}, a]) = [x_1, x_2, \ldots, x_{k-1}]$. Suppose $[x_1, x_2, \ldots, x_{k-1}, a]$ is even. The case $[x_1, x_2, \ldots, x_{k-1}, a]$ being odd is similar and will be omitted. By Lemma 4.9, ϕ either uniformly preserves or reverses parity of vertices in Q. We consider two subcases.

The first case is ϕ preserves parity, that is, $[x_1, x_2, \ldots, x_{k-1}]$ is even. Then the neighbours of $[x_1, x_2, \ldots, x_{k-1}, a]$ in Q via star-arcs are

$$\begin{gathered}
[x_1, x_2, x_3, \ldots, x_{k-1}, a] \rightarrow [x_2, x_1, x_3, \ldots, x_{k-1}, a], \\
[x_1, x_2, x_3, \ldots, x_{k-1}, a] \leftarrow [x_3, x_2, x_1, x_4 \ldots, x_{k-1}, a], \\
[x_1, x_2, x_3, \ldots, x_{k-1}, a] \rightarrow [x_4, x_2, x_3, x_1 \ldots, x_{k-1}, a], \\
\vdots \\
[x_1, x_2, x_3, \ldots, x_{k-1}, a] \leftarrow [x_{k-1}, x_2, x_3, \ldots, x_{k-2}, x_1, a].
\end{gathered}$$

The same rule is applied in $\overrightarrow{S_{n-1,k-1}}$ on the symbols $\{1, 2, \ldots, n\} \setminus \{a\}$, we have

$$\begin{gathered}
[x_1, x_2, x_3, \ldots, x_{k-1}] \rightarrow [x_2, x_1, x_3, \ldots, x_{k-1}], \\
[x_1, x_2, x_3, \ldots, x_{k-1}] \leftarrow [x_3, x_2, x_1, x_4 \ldots, x_{k-1}], \\
[x_1, x_2, x_3, \ldots, x_{k-1}] \rightarrow [x_4, x_2, x_3, x_1 \ldots, x_{k-1}], \\
\vdots \\
[x_1, x_2, x_3, \ldots, x_{k-1}] \leftarrow [x_{k-1}, x_2, x_3, \ldots, x_{k-2}, x_1].
\end{gathered}$$

Now for the residual-arcs, the rule depends on whether the arc belongs to a fundamental clique whose leading vertex is even or odd. We note that ϕ preserves the natural ordering of a fundamental clique. Since ϕ preserves parity, the corresponding fundamental cliques in Q and $\overrightarrow{S_{n-1,k-1}}$ are oriented the same way. Hence Q is isomorphic to $\overrightarrow{S_{n-1,k-1}}$ via ϕ.

The second case is ϕ reverses parity, that is, $[x_1, x_2, \ldots, x_{k-1}]$ is odd. Then the neighbours of $[x_1, x_2, \ldots, x_{k-1}, a]$ in Q via star-arcs are

$$
\begin{aligned}
[x_1, x_2, x_3, \ldots, x_{k-1}, a] &\longrightarrow [x_2, x_1, x_3, \ldots, x_{k-1}, a],\\
[x_1, x_2, x_3, \ldots, x_{k-1}, a] &\longleftarrow [x_3, x_2, x_1, x_4 \ldots, x_{k-1}, a],\\
[x_1, x_2, x_3, \ldots, x_{k-1}, a] &\longrightarrow [x_4, x_2, x_3, x_1 \ldots, x_{k-1}, a],\\
&\vdots\\
[x_1, x_2, x_3, \ldots, x_{k-1}, a] &\longleftarrow [x_{k-1}, x_2, x_3, \ldots, x_{k-2}, x_1, a].
\end{aligned}
$$

The same rule is applied in $\overrightarrow{S_{n-1,k-1}}$ on the symbols $\{1, 2, \ldots, n\} \setminus \{a\}$, we have

$$
\begin{aligned}
[x_1, x_2, x_3, \ldots, x_{k-1}] &\longleftarrow [x_2, x_1, x_3, \ldots, x_{k-1}],\\
[x_1, x_2, x_3, \ldots, x_{k-1}] &\longrightarrow [x_3, x_2, x_1, x_4 \ldots, x_{k-1}],\\
[x_1, x_2, x_3, \ldots, x_{k-1}] &\longleftarrow [x_4, x_2, x_3, x_1 \ldots, x_{k-1}],\\
&\vdots\\
[x_1, x_2, x_3, \ldots, x_{k-1}] &\longrightarrow [x_{k-1}, x_2, x_3, \ldots, x_{k-2}, x_1].
\end{aligned}
$$

Now for the residual-arcs, the rule depends on whether the arc belongs to a fundamental clique whose leading vertex is even or odd. We note that ϕ preserves the natural ordering of a fundamental clique. Since ϕ reverses parity, the corresponding fundamental cliques in Q and $\overrightarrow{S_{n-1,k-1}}$ are oriented in the opposite way. (Compare the left picture in Figure 4 and the right picture in Figure 4.) Hence Q is isomorphic to $\overleftarrow{S_{n-1,k-1}}$ via ϕ.

The case k being odd is similar. □

Theorem 4.12. *Suppose $k \geq 2$. Then $\overrightarrow{S_{n,k}}$ is $\lfloor \frac{n-1}{2} \rfloor$-connected.*

Proof. We will show this by induction on k. By Corollary 4.8, $\overrightarrow{S_{n,2}}$ is $\lfloor \frac{n-1}{2} \rfloor$-connected. So we may assume $k \geq 3$. Moreover, it is easy to see that $\overrightarrow{S_{4,3}}$ is strongly connected. Hence we may assume $n \geq 5$. Let H_i be the subgraph of $\overrightarrow{S_{n,k}}$ induced by vertices with i in the second position if $n-k+1$ is even and in the k-th position if $n-k+1$ is odd. Then it is isomorphic to either $\overrightarrow{S_{n-1,k-1}}$ or $\overleftarrow{S_{n-1,k-1}}$ by Lemma 4.10 and Lemma 4.11. In either case, H_i is $\lfloor \frac{n-2}{2} \rfloor$-connected. For any distinct i and j, there are $\frac{(n-2)!}{(n-k)!}$ independent star-arcs between

H_i and H_j. Such an arc has end-vertices of the form $[j, i, a_3, \ldots, a_k]$ and $[i, j, a_3, \ldots, a_k]$ if $n-k+1$ is even but the form $[j, a_2, a_3, \ldots, a_{k-1}, i]$ and $[i, a_2, a_3, \ldots, a_{k-1}, j]$ if $n-k+1$ is odd. Also it is easy to see that there are at least $\left\lfloor \frac{(n-2)!}{2(n-k)!} \right\rfloor$ arcs directed from H_i to H_j and vice versa. Now consider the deletion of a subset T of the vertices where $|T| = \left\lfloor \frac{n-1}{2} \right\rfloor - 1$. Let $T_i = T \cap H_i$ for each $i = 1, 2, \ldots, n$.

Suppose n is even, then $\lfloor \frac{n-1}{2} \rfloor - 1 = \frac{n-2}{2} - 1$. We already know that each H_i is $\lfloor \frac{n-2}{2} \rfloor = \frac{n-2}{2}$-connected, therefore $H_i \setminus T_i$ is strongly connected. We are done if we can show that there is at least one arc from $H_i \setminus T_i$ to $H_j \setminus T_j$ for every pair $i \neq j$. We will show that the number of (independent) arcs from H_i to H_j is greater than $\left(\lfloor \frac{n-1}{2} \rfloor - 1\right)$. Since $k \geq 3$ and n is even,

$$\left\lfloor \frac{(n-2)!}{2(n-k)!} \right\rfloor - \left(\frac{n-2}{2} - 1 \right) = \frac{(n-2)!}{2(n-k)!} - \left(\frac{n-2}{2} - 1 \right) \geq \frac{(n-2)!}{2(n-3)!} - \left(\frac{n-2}{2} - 1 \right)$$
$$\geq \frac{n-2}{2} - \left(\frac{n-2}{2} - 1 \right) = 1$$

Therefore, we are done if n is even.

Now we will examine the case when n is odd. In this case $n = 2p+1$ for some p and $\lfloor \frac{n-1}{2} \rfloor - 1 = p - 1$ and H_i is $\lfloor \frac{n-2}{2} \rfloor = \lfloor \frac{2p-1}{2} \rfloor = (p-1)$-connected. Therefore $H_i \setminus T_i$ is strongly connected if $|T_i| \leq p-2$.

Case 1: $|T_i| \leq p-2$ for all i.

We are done if $\left\lfloor \frac{(n-2)!}{2(n-k)!} \right\rfloor - 1 \geq (p-1)$. If $k \geq 4$ then $\left\lfloor \frac{(n-2)!}{2(n-k)!} \right\rfloor = \frac{(n-2)!}{2(n-k)!}$, so

$$\frac{(n-2)!}{2(n-k)!} - (p-1) \geq \frac{(n-2)!}{2(n-4)!} - (p-1) \geq \frac{(2p-1)(2p-2)}{2} - (p-1)$$
$$= \frac{(2p-1)(2p-2)}{2} - (p-1) \geq 1$$

since $p \geq 2$ as $n \geq 5$. Hence there is an arc from $H_i \setminus T_i$ and $H_j \setminus T_j$ and vice versa. Moreover, each $H_i \setminus T_i$ is strongly connected so $\overrightarrow{S_{n,k}} \setminus T$ is strongly connected. However, if $k = 3$ then

$$\left\lfloor \frac{(n-2)!}{2(n-k)!} \right\rfloor - (p-1) = \left\lfloor \frac{(n-2)!}{2(n-3)!} \right\rfloor - (p-1)$$
$$= \left\lfloor \frac{(2p-1)}{2} \right\rfloor - (p-1) \geq (p-1) - (p-1) = 0$$

which gives us the possibility that there are no arcs between subgraphs $H_i \setminus T_i$ and $H_j \setminus T_j$. This situation will only occur when $T = T_i \cup T_j$. Hence we may assume $T = T_1 \cup T_2$. Since the other H_i's are undamaged, the digraph induced by the vertices in them is strongly connected. Hence there is an arc from $H_1 \setminus T_1$ ($H_2 \setminus T_2$) to H_3 and vice versa. Therefore, $\overrightarrow{S_{n,k}} \setminus T$ is strongly connected.

Case 2: $|T_1| = p - 1$ and $|T_i| = 0$ for all $i \neq 1$.

Clearly the digraph X induced by the union of the vertices in $H_2, H_3, \ldots, H_n$ is strongly connected. Let Y be the strong component of $\overrightarrow{S_{n,k}} \setminus T$ containing X. Let C be a strongly connected component of $H_1 \setminus T_1$.

Case 2a: C is not a single isolated vertex.

If C contains a star-arc then we are done. To see this, recall that if uv is a star-arc, then its end-vertices have opposite parity. Hence the two arcs between $\{u, v\}$ and X must be in opposite directions. (Recall that the star-arcs between the H_i's are oriented 2-edges if $n-k+1$ is even and oriented k-edges if $n-k+1$ is odd. Therefore these arcs are either all directed from an even vertex to an odd vertex or all directed from an odd vertex to an even vertex.) Hence C is part of Y. Suppose C contains no star-arc. Then C must be a subgraph of a (directed) fundamental clique of H_1. We know that a fundamental clique is of size $(n-1)-(k-1)+1 = n-k+1$. At this point it is enough to show that C contains an odd vertex and an even vertex, because they will give arcs between C and X in both directions through star-arcs. By Proposition 4.3, a fundamental clique is isomorphic to $\overleftarrow{K_{n-k+1}}$ or $\overrightarrow{K_{n-k+1}}$. Since C is a subgraph of a directed fundamental clique, if it were to contain all odd or all even vertices then it would not be strongly connected by the definition of the orientation. So it must contain at least one vertex of each parity.

Case 2b: C is an isolated vertex, π_1.

We will show that there is a directed path containing π_1 in $\overrightarrow{S_{n,k}} \setminus T$ that starts and ends in X. Assume that the arc connecting π_1 to Y is directed from π_1 to a vertex x in X. Within H_1, π_1 has p arcs directed into it and $p-1$ arcs leaving it. Therefore there must exist an arc entering π_1 from another vertex π_2 in $H_1 \setminus T_1$. If there is an arc directed from a vertex in X to π_2 then we are done because we have a path from X to π_2 to π_1 to X. If not, we may show by the same reasoning that π_2 must also have an arc entering it in $H_1 \setminus T_1$. Since at least one arc is entering π_2 in $H_1 \setminus T_1$, we can find π_3 in H_1 where there is an arc from π_3 to π_2 in $H_1 \setminus T_1$. Again, if there is an arc from a vertex in X directed to π_3 we are done. If not then we may repeat this process and eventually either find a vertex π_m with an arc from X leading into it in which case we will be done, or we will continue to grow this path and it will eventually generate a directed cycle $\pi_k \pi_{k-1} \dots \pi_m$ since H_1 has finitely many vertices. If we find such a cycle, this gives us a non-singleton strongly connected component in $H_1 \setminus T_1$, which we have previously shown must be part of Y. Thus there exists a path from Y into this cycle, from which there exists a directed path to π_1 by construction, and hence π_1 is part of Y. Above we assumed that the arc connecting π_1 to X was directed from π_1 to a vertex in X, if it is the other way around and the arc connecting π_1 to X is directed from a vertex in X to π_1, then the exact same argument with all the directions reversed can be used to show π_1 is part of Y.

□

5. Conclusion

In this paper we have shown that $\overrightarrow{S_{n,k}}$ is $\lfloor \frac{n-1}{2} \rfloor$-connected and hence maximally connected. This reinforces the knowledge that $\overrightarrow{S_{n,k}}$ is a good generalization of the unidirectional star graphs and that its definition is the best possible in terms of connectivity.

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